Cocategory and Nilpotency
of homotopy self equivalence
New research subject

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My ISMAIL MAMOUNI
CPA-CRMEF Rabat

Plan

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I.2: Cocategory
I.3: Self equivalences
I.4: Derivation

II. Research subject

II.1: Position of the problem
II.2: Algebrization
II.3: Direction of research

I - Materials

I.1) Nilpotency (rational)

Def 1 A : algebra

$nil(A) := \max\{k \in \mathbb{N}$ st $a^k \neq 0$ $\forall a \in i\}$

Def 2 :

X simply connected top space

$nil_0(X) :=$ least k st there exists a dg-model of X of nilpotency equal to k

Rmk : $nil_0(X)$ rational homotopic invariant

I.2) Cocategory (rational)

Def 1 : $(\mathbb{L}_V, \partial)$ minimal Quillen model

$\mathrm{cocat}(\mathbb{L}_V) = \min \{ k$ st $\mathbb{L}_V \to \mathbb{L}_V / \mathbb{L}^{\geq k+1} \}$

admits a retraction

if such k dont exists we put $\mathrm{cocat}(\mathbb{L}_V) = \infty$

Def 2

S X simply connected top space of Quillen minimal model $(\mathbb{L}_V, \partial)$

we put $\mathrm{cocat}_0(X) := \mathrm{cocat}(\mathbb{L}_V)$

Rmk : $\mathrm{cocat}_0(X)$ is a rational homotopical invariant

I-3) Self equivalences.

Def: X pointed top space and $f: X \to X$

f homotopy self equiv $\Leftrightarrow$ f admits an homotopy inverse

$\Leftrightarrow$ f is isomorphism of X in the pointed homotopy category

Notation

$aut(X) := \{$ homotopy self equiv of X $\}$
$aut_0(X) := \{$ rational homotopy self equivalences $\}$

Useful Remarks

- $aut(X)$ is a top subspace of $map(X, X)$ equipped with the weak topology
- $aut(X)$ is in general disconnected but all its components are of the same rational homotopy type
- We focus on the path component of the identity which will be confounded with $aut(X)$ and $aut_0(X)$

I-4) Derivation

Def (A, d) dga and $\theta : A \to A$

θ derivation of degree n if $\theta : A^* \to A^{*-n}$

$\theta(ab) = \theta(a)b - (-1)^{n|a|} a\,\theta(b)$

Notations

$Der^n(A) = \{$ derivation of A of degree $n \}$

$Der^*(A) = \bigoplus Der^n(A)$

$Der_B^*(A) = \{ \theta \in Der^*(A)$ st $\theta = 0$ over $B(A) \}$

Thm $Der^*(A)$ is dgla

with $[\theta_1, \theta_2] = \theta_1\theta_2 - (-1)^{|\theta_1||\theta_2|}\theta_2\theta_1$

$D(\theta) = [d, \theta]$

II - Research Subject

II.1) Position of the problem

If X simply connected we ask the following

$$\mathrm{cocat}_0(B\mathrm{aut}(X)) < \infty \Rightarrow \mathrm{nil}_0(\mathrm{aut}(X)) < \infty$$

II.2) Algebrization of the problem

Thm 1 : Felix, Lupton, Smith (HHA, 2010)

If X simply connected finite cw-complex then

$$\pi_*(\mathrm{aut}(X)) \otimes \mathbb{Q} \cong H_*(\mathrm{Der}(\Lambda V, D))$$

where $(\Lambda V, d)$ a minimal Sullivan model of X

Thm 2 Dold, Lashof (Illinois Journal, 59)

aut(X) admits a classifying space Baut(X) which is also a CW-complex and simply connected. Thus Baut(X) admits a rationalization

Thm 3

if X is a finite and simply connected CW complex ~~then~~ of $(\Lambda V, d)$ as Sullivan minimal model then $(\mathrm{Der}(\Lambda V), D)$ is a minimal Quillen model of Baut(X)

Open Problem (algebraic version)

We consider $(\Lambda V, d)$ a minimal Sullivan model and ask the following

$$\mathrm{cocat}\,(\mathrm{Der}(\Lambda V), D) < \infty \overset{?}{\Longrightarrow} \mathrm{nil}\,(H^*(\mathrm{Der}(\Lambda V), D)) < \infty$$

II.3) Directions of research (Spain thesis 1984)

Thm 1 if X is formal then

$\text{cocat}_0(X) \leq 2$ or $\text{cocat}_0(X) = \infty$

$\text{cocat}_0(X) = \mathcal{E}_0(X)$

where $\mathcal{E}_0(X) :=$ least k of $\mathbb{L}_V \to \mathbb{L}_V / \mathbb{L}_V^{\geq k+1}$ induces injection in homology | dual

Thm 2

Rmk : $\mathcal{E}_0$ rational homotopical invariant dual to that of Toomer

Thm 2 if X simply connected and pointed top space of finite type

then

$\boxed{\text{nil}(\pi_{*-1}(X) \otimes \mathbb{Q}) \leq \mathcal{E}(X) \leq \text{cocat}_0(X)}$

all invariant are equal when X coformal

$\boxed{\text{cocat}(X_{\mathbb{Q}}) \leq \text{cocat}_0(X)}$ | equality when X coformal

Rmk $\text{cocat}(X_{\mathbb{Q}})$: in the sense of Ganea

Thm 3 $\text{cocat}_0(X) = 0 \iff X$ contractible

$\text{cocat}_0(X) = 1 \iff X \sim_{\mathbb{Q}} K(\pi, n)$

$\text{cocat}_0(X) < \infty \implies \pi_k(X) = 0$ for a certain $k \geq N$