ENTROPY BY UNIT LENGTH FOR THE GINZBURG-LANDAU EQUATION ON THE LINE. A HILBERT SPACE FRAMEWORK.

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Abstract. It is well-known that the Ginzburg-Landau equation on $\mathbb{R}$ has a global attractor [15] that attracts in $L^\infty_{\text{loc}}(\mathbb{R})$ all the trajectories. This attractor contains bounded trajectories that are analytical functions in space. A famous theorem due to P. Collet and JP. Eckmann asserts that the ε-entropy per unit length in L^∞ of this global attractor is finite and is smaller than the corresponding complexity for the space of functions which are analytical in a strip. This means that the global attractor is flatter than expected. We explain in this article how to establish the Collet-Eckmann Theorem in a Hilbert space framework.

1. Introduction.

1.1. A tribute to Collet and Eckmann.

This article, which partakes of partial differential equations and ergodic theory, is concerned with some properties of global attracting sets for the dynamical systems provided by the complex Ginzburg-Landau equation (CGL) on the whole line $x \in \mathbb{R}$

$$\partial_t u = (1 + i\alpha)\Delta u + u - (1 + i\beta)u|u|^2. \tag{1}$$

Here the unknowns u map $\mathbb{R}^+_t \times \mathbb{R}_x$ into $\mathbb{C}$ and α, β are parameters that belong to $\mathbb{R}$. As we shall see in the sequel, estimating the size of global attracting sets for (1) provides us with some extra difficulties that come from the fact that $\mathbb{R}$ equipped with the Lebesgue measure has infinite volume.

In a seminal article P. Collet and JP. Eckmann [5] have proved that the global attractor for (1) has finite complexity, using the *ε-entropy by unit length* as in [13]; this is required since the global attractor has infinite dimension. On the other hand, since the CGL equation (1) is of parabolic type, then one can prove that the global attractor for this dynamical system is included into some subset of analytical functions in a neighboring strip of the real axis. The Collet-Eckmann result proves that in fact the ε-entropy by unit length of this global attractor is much smaller than the corresponding dimension for this set of analytical functions. The proof of Collet and Eckmann use intensively the L^∞ norm. Here we would like to prove the Collet-Eckmann theorem in a Hilbert space framework (namely L^2); we believe that this framework is more tractable for a larger class of PDEs than those of parabolic type.

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To study the CGL equation on the whole line on global Sobolev spaces as $L^2(\mathbb{R})$ or $H^1(\mathbb{R})$ is somehow irrelevant for the physics. Actually these spaces do not contain structures as fronts or periodic solutions in space. Therefore in the last decades a particular attention in solving (1) in local spaces has increased, going back to [4], [9] and [10] for the initial value problem. For the dynamical properties of this equation, we already have pointed out the Collet-Eckmann study of these extensive dynamical systems. Another approach, due to S. Zelik, using weighted Sobolev spaces has been advocated in [20], [21] for the study of extended dynamical systems as reaction-diffusion equations on the whole space.

This article is organized as follows; we complete this introduction by recalling some well known facts about the dynamics of the solutions to the CGL. To begin with, we make precise the notion of attractor. Then we will introduce the ε-entropy by unit length. We then state our main result that compares with the Collet-Eckmann result.

We conclude this subsection by introducing some notations. For a given Hilbert space as $L^2(B)$ wherein B is an interval included in $\mathbb{R}$, the scalar product of two functions reads $\mathrm{Re}\int_B u\bar{v}dx$. Throughout this article we will use constants denoted by C, C_1, K, ... that may vary from one line to one another, and that may depend on the data α, β. To compare two functions we write $f \preceq g$ if there exists a numerical constant C such that $f \leq Cg$. In other words this relation reads also $f = O(g)$ with the Landau notations. We set $f \simeq g$ if $f = O(g)$ and $g = O(f)$.

1.2. **Attractor for the CGL on the line.** To begin with, let us observe that the equation (1) is translation invariant, i.e that for any u solution to (1) then

$$T_y u(x) = u(x - y),$$

is also solution to the equation (here the time variable t is omitted). This implies a lack of compactness for the trajectories in classical Sobolev spaces and makes the dynamical study of the CGL equation different. To overcome this difficulty, we use the framework of *uniformly local spaces*, as in the article [15], and the notion of Z-Z_ρ attractor as in [7]. For the theory of infinite-dimensional dynamical systems we would like to refer to [1], [2], [11], [16] and [17]; we follow here the framework described in [16].

Consider $(Z, \|.\|)$ a Banach space, or more generally a Fréchet space, that is continuously embedded into another space $(Z_\rho, \|.\|_\rho)$. Assume that the IVP problem associated with (1), supplemented with initial data in Z, is globally well-posed. Hence we have a semigroup $S_t : Z \to Z, u_0 \mapsto u(t)$. We then recall

Definition 1.1 ((Z, Z_ρ)-Attractor)**.** *A set $\mathcal{A}$ is called (Z, Z_ρ)-Attractor for S_t in Z if the following conditions hold:*

1. *$\mathcal{A}$ is nonempty, closed, bounded in Z, and compact in Z_ρ.*
2. *$\mathcal{A}$ is invariant under S_t, i.e., $S_t(\mathcal{A}) = \mathcal{A}$ for all $t > 0$.*
3. *Every $B \subset \mathcal{A}$ which is bounded in Z is attracted to $\mathcal{A}$ in Z_ρ as follows,*

$$dist_{Z_\rho}(S_t(B), \mathcal{A}) := \sup_{b\in\mathcal{B}} \inf_{a\in\mathcal{A}} \|S_t(b) - a\|_\rho \to 0 \quad for \quad t \to \infty \ .$$

We now recall from [15] one framework to study CGL equations on the whole line. The functional spaces are chosen with some weight functions in order to enforce some compactness, and large enough to include all physically relevant solutions as periodic solutions, fronts, which have no decay if x goes to the infinity. First we introduce a positive weight function $\rho : \mathbb{R} \to (0, \infty)$ which is continuous, bounded, and such that $\int_{\mathbb{R}} \rho(x)dx < +\infty$. We also assume

$$\max(|\rho'(x)|, |\rho''(x)|) \leq \rho(x).$$

For instance $\rho(x) = (1 + x^4)^{-1}$ works and we shall deal with this weight function in the sequel. Introduce then

$$\|u\|^2_{L^2_{\mathrm{ul}}(\mathbb{R})} = \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} \rho(x-y)|u(x)|^2 dx,$$

and $L^2_{\mathrm{ul}}(\mathbb{R}) = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}); \|u\|_{L^2_{\mathrm{ul}}} < \infty$ and $\|T_y u - u\|_{L^2_{\mathrm{ul}}} \to 0$ *as* $y \to 0\}$. These uniformly local spaces admit Sobolev versions as $H^1_{\mathrm{ul}}(\mathbb{R}) = \{u \in L^2_{\mathrm{ul}}(\mathbb{R}); u_x \in L^2_{\mathrm{ul}}(\mathbb{R})\}$. Then let H_ρ denote the space of function

$$H_\rho = \{u \in H^1_{\mathrm{loc}}(\mathbb{R}); \int_{\mathbb{R}} (|u(x)|^2 + |u_x(x)|^2)\rho(x)dx < +\infty\}.$$

We now recall from [15]

Theorem 1.1. *The IVP for (CGL) is well defined in $H^1_{\mathrm{ul}}(\mathbb{R})$; moreover for any $t > 0$ and $n \in \mathbb{N}$, S_t maps any given bounded set $B \subset H^1_{\mathrm{ul}}(\mathbb{R})$ into a bounded set in $H^n_{\mathrm{ul}}(\mathbb{R})$. Furthermore (1) has a global $(H^1_{\mathrm{ul}}, H^1_\rho)$-attractor $\mathcal{A}$ which is invariant by translation.*

Proof. See [15]; see also [18] for the proof of analyticity for the functions that belong to the global attractor. □

Remark 1.1. *The attractor $\mathcal{A}$ will not depend on the particular choice of the weight function ρ.*

1.3. **ε-Entropy per Unit Length.** We follow here Collet and Eckmann [5]. To begin with, we recall from [8] that the CGL equation for $x \in [-L, L]$ with periodic boundary conditions has a global attractor whose fractal dimension is finite. The fractal or box-dimension is defined as follows. Consider $N'(\varepsilon)$ the minimal number of balls of radius ε in some Banach space X, say $L^\infty(\mathbb{R})$, to cover $\mathcal{A}$. Then

$$dim_F \mathcal{A} = \lim_{\varepsilon\to 0} \frac{\log(N'(\varepsilon))}{-\log(\varepsilon)}. \tag{2}$$

Actually, in [8] it is given a lower bound for this dimension as $C(\alpha, \beta)L$, where $C(\alpha, \beta)$ is a constant which depends on the data for the equation. As a consequence, since this periodic attractor is embedded into the attractor defined in the previous section, letting $L \to +\infty$ we prove that the attractor for the CGL equation on the line has infinite dimension.

The idea is then to compute the complexity of the attractor through a window of size $2L$. Consider $\mathcal{A}$ and consider $\mathcal{A}/Q_L$ the set of the restrictions of the functions in the attractor to a space interval Q_L of width $2L$. Unfortunately, it turns out that the fractal and Hausdorff dimension in L^∞ of $\mathcal{A}/Q_L$ are also infinite.

This example shows that, in contrast to bounded domains, we cannot expect any finite dimensional reduction in general and the dynamics reduced to the global attractor remains infinite dimensional. On the other hand, intuitively the global attractor is thinner (or flatter) than the whole space and we need another tool to express that the dynamics on the attractor is finite-dimensional. Then the idea is to introduce the ε-entropy per unit length see [13], as in [19], [20]. Let N'_{Q_L}

the minimum number of balls of radius ε which are necessary to cover $\mathcal{A}|_{Q_L}$ in $L^\infty(Q_L)$(this number is finite).

Definition 1.2. *(ε-entropy by unit lenght in L^∞)*

$$H'_\varepsilon = \lim_{L\to\infty} \frac{\log(N'_{Q_L}(\varepsilon))}{2L}.$$

Then it is proven in [5]

Theorem 1.2. *There exists C a constant that depends on the data α, β of the equation such that for ε small*

$$\frac{1}{C}\log(\frac{1}{\varepsilon}) \leq H'_\varepsilon \leq C\log(\frac{1}{\varepsilon}).$$

In the remaining of this article we prove that we can substitute a suitable L^2 local norm to the L^∞ norm.

1.4. **Our main result.** To begin with, we define the local L^2 norm that will be used throughout this article. Consider B an interval of $\mathbb{R}$. Let $|B|$ denotes the length of B. For any function u that is locally square integrable on $\mathbb{R}$, we state

Definition 1.3.

$$||u||^2_{L^2(B)} = \frac{1}{|B|}\int_B |u(x)|^2 dx.$$

For a large L define $Q_L = [-L, L]$ and $N_{Q_L}(\varepsilon)$ the minimum number of balls of radius ε which are necessary to cover $\mathcal{A}|_{Q_L}$ in $L^2(Q_L)$(this number is finite). Then define

$$H_\varepsilon = \lim_{L\to\infty} \frac{\log(N_{Q_L}(\varepsilon))}{2L}. \tag{3}$$

Then

Theorem 1.3. *There exists C a constant that depends on the data α, β of the equation such that for ε small*

$$\frac{1}{C}\log(\frac{1}{\varepsilon}) \leq H_\varepsilon \leq C\log(\frac{1}{\varepsilon}).$$

The remaining of the article is organized as follows. In the next section, we prove the *existence* of the entropy by unit length with our scaled L^2 norm. Then we move to the proof of Theorem 1.3. Actually due to the inequality $||u||_{L^2(B)} \leq ||u||_{L^\infty}$ the upper bound for the L^2 norm is a consequence of the Collet-Eckmann result. Nevertheless, we indicate in the Section 3 a self-contained proof which has in own interest. For the lower bound, the method in Collet-Eckmann does not apply straightforwardly, neither the construction of the unstable manifold as in [20]; hence we provide a complete proof in Section 4.

2. Existence of the ε-entropy by unit length.

We establish here that the ε-entropy H_ε exists in $\mathbb{R}^+ \cup \{+\infty\}$. To begin with, we state and prove

Lemma 2.1. *Let B and B' denote two disjoint bounded sets of $\mathbb{R}$. We denote by $N_B(\varepsilon)$ and $N_{B'}(\varepsilon)$ the minimum number of balls in $L^2(B)$ and in $L^2(B')$ of radius ε which is needed to cover respectively $\mathcal{A}|_B$ and $\mathcal{A}|_{B'}$. We have*

$$N_{B\cup B'}(\varepsilon) \leq N_B(\varepsilon)N_{B'}(\varepsilon).$$

Remark 2.1. *Actually, we can relax the assumption B and B' disjoints. We observe that the result holds true if the Lebesgue measure of $B \cap B'$ is 0.*

Proof. We denote by $u|_B$ the restriction of u to B . Introduce $u^B_{i=1,2..N_B(\varepsilon)}$ and $u^{B'}_{j=1,2..N_{B'}(\varepsilon)}$ the centers of the balls in $L^2(B)$ and in $L^2(B')$ of radius ε to cover respectively $\mathcal{A}|_B$ and $\mathcal{A}|_{B'}$. Introduce the functions

$$u^{B\cup B'}_{i,j} = \begin{cases} u^B_i \ for\ x \in B, \\ u^{B'}_j \ for\ x \in B'. \end{cases}$$

We prove below that the balls centered at $u^{B\cup B'}_{i,j}$ cover $\mathcal{A}|_{B\cup B'}$. Let $u \in \mathcal{A}$ and pick u^B_i, $u^{B'}_j$ the functions such that $\|u|_B - u^B_i\|_{L^2(B)} \le \varepsilon$ and $\|u|_{B'} - u^{B'}_j\|_{L^2(B')} \le \varepsilon$. Then we have

$$\begin{aligned} \|u|_{B\cup B'} - u^{B\cup B'}_{i,j}\|^2_{L^2(B\cup B')} &= \frac{1}{|B\cup B'|}\int_{B\cup B'} |u - u^{B\cup B'}_{i,j}|^2 dx \\ &= \frac{1}{|B\cup B'|}(\int_B |u - u^B_i|^2 dx + \int_{B'} |u - u^{B'}_j|^2 dx) \\ &= \max(\frac{1}{|B|}\int_B |u - u^B_i|^2 dx, \frac{1}{|B'|}\int_{B'} |u - u^{B'}_j|^2 dx), \end{aligned}$$

since $|B| + |B'| = |B\cup B'|$ because $B\cap B' = \emptyset$. Then we have $\|u|_{B\cup B'} - u^{B\cup B'}_{i,j}\|_{L^2(B\cup B')} \le \varepsilon$ and the result follows promptly. $\square$

Proposition 2.1. *The limit $H_\varepsilon = \lim_{L\to\infty} \frac{log(N_{Q_L}(\varepsilon))}{2L}$ exists.*

Proof. To begin with, let us observe that the global attractor is invariant by translation. Then if Q_L is any interval of width $2L$, $N_{Q_L}(\varepsilon)$ depends on L but is independent of the center of the interval Q_L. Consider now an interval of width $2(L+L')$, say $[-L-L', L+L']$. We can split this interval into two closed intervals Q_L and $Q_{L'}$ whose length are respectively $2L$ and $2L'$ and such that $Q_L \cap Q_{L'}$ is one point in $\mathbb{R}$. Clearly, we cover $\mathcal{A}/Q_L$ and $\mathcal{A}/]-L, L[$ by the same number of balls and then by the previous lemma

$$a(L+L') = logN_{Q_{L+L'}}(\varepsilon) \le a(L) + a(L'). \tag{4}$$

Then the sequence $a(L)$ is a van Hove sequence and the convergence of $a(L)L^{-1}$ if $L \to +\infty$ is standard; consider

$$m = [\frac{L}{L'}] = \max\{k \in \mathbb{N}; kL' \le L\}.$$

Set $L = kL' + r$. Then, by induction on k

$$a(L) \le ka(L') + a(r). \tag{5}$$

Then divide by L and let L go to the infinity (for a fixed L', then $k \to +\infty$). to obtain

$$\limsup_{L\to+\infty} \frac{a(L)}{L} \le \frac{a(L')}{L'}. \tag{6}$$

We infer from the previous inequality that

$$\limsup_{L\to+\infty}\frac{a(L)}{L}\le\liminf_{L'\to+\infty}\frac{a(L')}{L'}. \tag{7}$$

and the proof is completed. □

3. **Proof of the upper bound in Theorem 1.3.** The next two subsections are concerned with the proof of two key lemmas in the proof of the Theorem. The first one describes how the semigroup S_t deforms the balls. The second one is a covering lemma. We conclude the proof of the upper bound in a third subsection.

3.1. **Deformation of the balls.** For later use we describe how the semigroup S_t of CGL acts on balls in $\mathcal{A}_{Q_L}\subset L^2(Q_L)$, for short times (say $t\in[0,1]$). Consider two functions u_0 and v_0 in $\mathcal{A}$. Then the trajectories S_tu_0 and S_tv_0 remain in a bounded set of $W^{1,+\infty}(\mathbb{R})$ (we shall use this statement in the sequel without notice). We now prove

Proposition 3.1. *Set $w_0=v_0-u_0$. Let $\varepsilon>0$ with $L>\frac{2}{\varepsilon}$. Assume*

$$\|w_0\|_{L^2(Q_L)}\le\varepsilon.$$

Then there exist a constant $c(\alpha,\beta)$ which depends on the data for the equation such that

$$\|S_1u_0-S_1v_0\|_{L^2_x(Q_{L-\frac{1}{\varepsilon}})}\le c(\alpha,\beta)\varepsilon, \tag{8}$$

and

$$\|\nabla(S_1u_0-S_1v_0)\|_{L^2_x(Q_{L-\frac{1}{\varepsilon}})}\le c(\alpha,\beta)\varepsilon. \tag{9}$$

Proof. This deformation Lemma appears in [5]. Here we proceed to computations using scalar products instead of properties of the kernel associated to the Ginzburg-Landau linear flow, which is very similar to the heat flow. To begin with we prove some energy estimates involving weighted norms. Introduce $w(t)=S_tv_0-S_tu_0$. There are two bounded functions $a=1-(1+i\beta)(S_tu_0+S_tv_0)\overline{S_tu_0}$ and $b=-(1+i\beta)(S_tv_0)^2$ such that

$$\partial_tw=(1+i\alpha)\Delta w+aw+b\overline{w}. \tag{10}$$

Consider the inner product of (10) with ρw. Then we have for a constant K that depends on the $W^{1,\infty}$ bounds on the attractor (then on the data of the equation) and that may vary from one line to one another, such that

$$\frac{1}{2}\frac{d}{dt}||w||^2_\rho+||w_x||^2_\rho\le(1+|\alpha|)\int_{\mathbb{R}}|\rho'||w||w_x|dx+K||w||^2_\rho. \tag{11}$$

Using $|\rho'|\le\rho$ and Young inequality we thus obtain

$$\frac{d}{dt}||w||^2_\rho+||w_x||^2_\rho\le K||w||^2_\rho. \tag{12}$$

Consider now the inner product of (10) with $-\rho\Delta w$ to obtain

$$\frac{1}{2}\frac{d}{dt}||w_x||^2_\rho+||w_{xx}||^2_\rho\le(1+|\alpha|)\int_{\mathbb{R}}|\rho'||w_{xx}||w_x|dx+K(||w_x||^2_\rho+||w||^2_\rho). \tag{13}$$

We then infer from this inequality, dropping some unnecessary terms

$$\frac{d}{dt}||w_x||^2_\rho\le K(||w_x||^2_\rho+||w||^2_\rho). \tag{14}$$

From (12) using Gronwall inequality we obtain for $t=1$

$$||w(1)||_\rho^2 \le e^K ||w_0||_\rho^2. \tag{15}$$

From (13) we obtain for $0 \le s \le 1$

$$||w_x(1)||_\rho^2 e^{-K} \le ||w_x(s)||_\rho^2 e^{-Ks} + K\int_s^1 e^{-K\tau} ||w(\tau)||_\rho^2 d\tau. \tag{16}$$

From (12) and (15) we also have

$$\int_0^1 ||w(\tau)||_\rho^2 d\tau \le K||w_0||_\rho^2.$$

We integrate (16) between 0 and 1, and using (16) we obtain

$$||w_x(1)||_\rho^2 \le K||w_0||_\rho^2. \tag{17}$$

These computations are standard for parabolic equations. It is worth pointing out that the computations remain valid if we substitute $\rho(. - y)$ to ρ for any given y. We now move to this weighted estimates to the local ones. On the one hand, there exists $\alpha > 0$ such that for $x \in (-\alpha, \alpha)$ then $\rho(x) \ge \frac{1}{2}$. Then for $l = L - \frac{1}{\varepsilon}$ there exists $m = O(l)$ points y_j in $[-l, l] \cap \alpha\mathbb{Z}$ such that such that for any x in $[-l, l]$,

$$\frac{1}{2} \le \sum_{1\le j\le m} \rho(x - y_j).$$

We then have

$$\frac{1}{2l}\int_{Q_l} |w(1,x)|^2 dx \le \frac{1}{l}\sum_{j=1}^m \int_{\mathbb{R}} \rho(x-y_j)|w(1,x)|^2 dx. \tag{18}$$

We infer from (15) that

$$\frac{1}{2l}\int_{Q_l} |w(1,x)|^2 dx \le \frac{K}{l}\sum_{j=1}^m \int_{\mathbb{R}} \rho(x-y_j)|w_0(x)|^2 dx. \tag{19}$$

To provide an upper bound on the right hand side of (19) we divide the integral according to the cases $|x| \le L$ and $|x| > L$. For the former case, we have

$$\frac{K}{l}\sum_{j=1}^m \int_{Q_L} \rho(x-y_j)|w_0(x)|^2 dx \le \frac{K||\sum_{j=1}^m \rho(. - y_j)||_{L^\infty}}{l}\int_{Q_L} |w_0(x)|^2 dx \le K\varepsilon^2. \tag{20}$$

Here we have used that $\rho(x) = (1+x^4)^{-1}$ has enough decay at the infinity. For the later case, we have

$$\frac{K}{l}\sum_{j=1}^m \int_{|x|\ge L} \rho(x-y_j)|w_0(x)|^2 dx \le K||w_0||_{L^\infty}^2 \max_{1\le j\le m}\int_{|x|\ge L} \rho(x-y_j)dx. \tag{21}$$

Here we use that if ρ has enough decay and since $L - l = \frac{1}{\varepsilon}$

$$\int_{|x|\ge L} \rho(x-y_j)dx \le c\int_L^{+\infty} \frac{dx}{(x-l)^4}dx = O(\varepsilon^2).$$

The same computations are valid for $w_x(1)$ using (17) instead of (15). We skip the details for the sake of conciseness and the proof of Proposition 3.1 is completed. □

3.2. **A covering lemma.** In the previous subsection, we have proven that a small ball in $L^2(Q_L)$ is mapped into some ball in $H^1(Q_l)$, for $l = L - \frac{1}{\varepsilon}$. The next statement computes the number of balls $L^2(Q_l)$ needed to cover the $H^1(Q_l)$ ball. This geometric statement is valid for any given $l > 0$.

Proposition 3.2. *Let Q_l be an interval in $\mathbb{R}$ of length $2l$, and assume that $\mathcal{U}$ is defined as follows*

$$\mathcal{U} = \Big\{ u : Q_l \mapsto \mathbb{C} \text{ such that } \|u\|_{L^2(Q_l)} \le a \text{ and } \|\nabla u\|_{L^2(Q_l)} \le b \Big\}$$

then one can cover $\mathcal{U}$ with $(\frac{2a}{\varepsilon} + 1)^{\frac{8bl}{\pi\varepsilon}}$ balls of radius ε in $L^2(Q_l)$.

Proof. Set $l = mr, m \in \mathbb{N}$, with r to be specified subsequently. We divide Q_l into m intervals $Q_i; 1 \le i \le m$ of length $|Q_i| = 2r$. We assume that $Q_i \cap Q_j$ has 0 Lebesgue measure if $i \ne j$.

Let us introduce the orthonormal family in $L^2(Q_l)$, $e_i(x) = \sqrt{\frac{l}{r}} \chi_{Q_i}(x)$, where

$$\chi_{Q_i}(x) = \begin{cases} 1 \ for \ x \in Q_i \\ 0 \ for \ x \in Q_l \setminus Q_i, \end{cases}$$

and denote by P the orthogonal projector onto the space spanned by $\{e_i\}_{i=1,2..m}$. Denoting by $(,)$ the scalar product in $L^2(Q_l)$, we can write

$$Pu(x) = \sum_{i=1}^{m} (u, e_i) e_i(x).$$

We now recall [3] the so-called Poincaré-Wirtinger inequality

$$\|u - \frac{1}{|Q_i|} \int_{Q_i} u(y)dy\|_{L^2(Q_i)} \le \frac{|Q_i|}{\pi} \|\nabla u\|_{L^2(Q_i)}. \tag{22}$$

Any function u in $H^1(Q_l)$ splits into $u = Pu + (u - Pu)$. To begin with we handle $u - Pu$ as follows

$$\begin{aligned} \frac{1}{2l} \int_{Q_l} |u(x) - Pu(x)|^2 dx &= \frac{1}{2l} \sum_{i=1}^{m} \int_{Q_i} |u(x) - Pu(x)|^2 dx \\ &= \frac{1}{2l} \sum_{i=1}^{m} \int_{Q_i} |u(x) - \frac{1}{2r} \int_{Q_i} u(y)dy|^2 dx \\ &\le \frac{4r^2}{2l\pi^2} \sum_{i=1}^{m} \int_{Q_i} |\nabla u(x)|^2 dx \\ &= \frac{4r^2}{2l\pi^2} \int_{Q_l} |\nabla u(x)|^2 dx \le \frac{4r^2 b^2}{\pi^2}. \end{aligned}$$

Then we fix r such that $\frac{2rb}{\pi} = \frac{\varepsilon}{2}$. Then on the one hand $m = \frac{l}{r} = \frac{4bl}{\varepsilon\pi}$, and on the other hand

$$\|u - Pu\|_{L^2(Q_l)} \le \frac{\varepsilon}{2}. \tag{23}$$

We now handle the Pu term. To begin with let us observe that

$$\|Pu\|_{L^2(Q_l)} = \sqrt{\sum_{i=1}^{m} |(u,e_i)|^2} \leq \|u\|_{L^2(Q_l)} \leq a. \tag{24}$$

We address the following issue: how many balls do we need to cover the ball $C_{2m} = \Big\{x \in \mathbb{R}^{2m}; \|x\| \leq a\Big\}$ in $PL^2(Q_l)$ which is isometric to $\mathbb{R}^{2m}$ equipped with the Euclidian norm $||.||$? For this purpose we use the following classical lemma

Lemma 3.1. *Let $\nu(\varepsilon)$ the minimum number of balls of radius ε to cover $C_{2m} = \Big\{x \in \mathbb{R}^{2m}; \|x\| \leq a\Big\}$. Then*

$$\nu(\varepsilon) \leq \Big(1 + \frac{2a}{\varepsilon}\Big)^{2m}.$$

Proof. Let $\nu_1(\varepsilon)$ the maximal number of points $y_1, y_2, ...y_s, ...$ belonging to C_{2m} such that $\|y_i - y_j\| > \varepsilon$ for $i \neq j$. To begin with we state and prove

$$\nu_1(\varepsilon) \leq \nu(\frac{\varepsilon}{2}) \leq \nu_1(\frac{\varepsilon}{2}). \tag{25}$$

To check that the left inequality in (25) is valid we proceed as follows: if $\nu_1(\varepsilon) > \nu(\frac{\varepsilon}{2})$, then by the pigeonhole principle there exist two points y_i and y_j $(i \neq j)$ in a same ball of radius $\frac{\varepsilon}{2}$; then $||y_j - y_i|| \leq \varepsilon$ and this is not valid according to the very definition of the y_is. We now prove the right inequality in (25), i.e $\nu(\varepsilon) \leq \nu_1(\varepsilon)$. Let y_j, $1 \leq j \leq \nu_1(\varepsilon)$ be defined as above. We observe that $\cup_j\{x; ||x - y_j|| \leq \varepsilon\}$ cover C_{2m}; if this last assertion is not valid then we can add another point $y_{\nu_1(\varepsilon)+1}$ to the family. Hence (25) is established.
To complete the proof of the lemma we observe that the balls of center y_j and or radii $\frac{\varepsilon}{2}$ are disjoints and included into the ball $\{x; ||x|| \leq a + \frac{\varepsilon}{2}\}$. Due to the very definition of the Lebesgue measure λ_{2m} we then have

$$\nu_1(\varepsilon)(\frac{\varepsilon}{2})^{2m}\lambda_{2m}(\{x; ||x|| \leq 1\}) \leq (a + \frac{\varepsilon}{2})^{2m}\lambda_{2m}(\{x; ||x|| \leq 1\}). \tag{26}$$

Then gathering (25) and (26) complete the proof of the lemma. □

We now complete this subsection. Consider here z_j the center of the balls in $\mathbb{R}^{2m} \simeq PL^2(Q_l)$ defined in the previous lemma. Then, for any u in $\mathcal{U}$ there exists one z_j such that

$$\|u - z_j\|_{L^2(Q_l)} \leq \|u - Pu\|_{L^2(Q_l)} + \|Pu - z_j\|_{L^2(Q_l)} \leq \varepsilon. \tag{27}$$

Then we cover $\mathcal{U}$ by $(1 + \frac{2a}{\varepsilon})^{\frac{8bl}{\pi\varepsilon}}$ balls. □

3.3. **Completing the proof of the main theorem.** We follow here closely [5]. For another proof see [14]. Our objective is to get an upper bound for $N_{Q_L}(\varepsilon)$. For this purpose we use the invariance of $\mathcal{A}$ by S_t to obtain a relation between $N_{Q_L}(\varepsilon)$ and $N_{Q_L}(\frac{\varepsilon}{2})$. Let us denote by $N_{Q_L}^{(t)}(\varepsilon)$ the number of balls B of radius ε needed to cover the set $S(t)\mathcal{A}|_{Q_L}$ in $L^2(Q_L)$ at time $t \geq 0$. We then state and prove

Lemma 3.2. *There exist constants C_1, C_2, C_3 that depend on the data of the equation but which are independent of L and of ε such that*

$$N_{Q_L}^{(t+1)}(\frac{\varepsilon}{2}) \leq N_{Q_L}^{t}(\varepsilon)C_1^L(\frac{C_2}{\varepsilon})^{\frac{C_3}{\varepsilon^2}}.$$

Proof. to begin with let us observe that the centers of the balls B of radius ε needed to cover the set $S(t)\mathcal{A}|_{Q_L}$ are not necessarily in $S(t)\mathcal{A}|_{Q_L}$. In any ball B we pick a point in $S(t)\mathcal{A}|_{Q_L}$ and thus we can cover $S(t)\mathcal{A}|_{Q_L}$ by $N^{(t)}_{Q_L}(\varepsilon)$ balls of radii 2ε whose *centers are in* $S(t)\mathcal{A}|_{Q_L}$. Denoting by $\mathcal{B}$ the set of these balls, using that $\mathcal{A}$ is invariant by S_t we then have

$$S_{t+1}\mathcal{A}|_{Q_L} \subset \bigcup_{B\in\mathcal{B}} S_1(B\cap S_t\mathcal{A})|_{Q_L}. \tag{28}$$

To cover $S_{t+1}\mathcal{A}|_{Q_L}$ by balls of radii $\frac{\varepsilon}{2}$ we will cover any set $S_1(B\cap S_t\mathcal{A})|_{Q_L}$ and then gather the bounds. To begin with we cover $S_1(B\cap S_t\mathcal{A})|_{Q_{L-\frac{1}{\varepsilon}}}$. Due to Proposition 3.1, this set is included into a ball in $H^1(Q_{L-\frac{1}{\varepsilon}})$ of radius $C(\alpha,\beta)\varepsilon$. We cover this set thanks to Proposition 3.2 by C_1^L balls of radius ε, where $C_1, C_2, ...$ denote constants that depend on the data α,β of the equation and that may vary from one line to one another. We now cover the set $S_1(B\cap S_t\mathcal{A})|_{Q_L-Q_{L-\frac{1}{\varepsilon}}}$ once again using Proposition 3.2 for a, b that are constants (independent of ε). Since $Q_L - Q_{L-\frac{1}{\varepsilon}}$ is the union of two intervals of length $\frac{1}{\varepsilon}$, we can cover the set $S_1(B\cap S_t\mathcal{A})|_{Q_L-Q_{L-\frac{1}{\varepsilon}}}$ by $(\frac{C_2}{\varepsilon})^{\frac{C_3}{\varepsilon^2}}$. Using Proposition 2.1 we cover any set $S_1(B\cap S_t\mathcal{A})|_{Q_L}$ by $C_1^L(\frac{C_2}{\varepsilon})^{\frac{C_3}{\varepsilon^2}}$ balls. □

We now introduce $n_*+1 \sim \log\frac{1}{\varepsilon}$. At time $t=0$ we need C_0^L balls of radius 1 to cover the set $\mathcal{A}$. We compute recursively the number of balls of radii $\frac{1}{2^n}$ to cover $S_{n+1}\mathcal{A}|_{Q_L}$, thanks to Lemma 3.2, as

$$N^{(n+1)}_{Q_L}(\frac{1}{2^{n+1}}) \le C_1^L(C_2 2^n)^{C_3 4^n} N^n_{Q_L}. \tag{29}$$

Then

$$N^{(n+1)}_{Q_L}(\frac{1}{2^{n+1}}) \le C_0^L C_1^{nL}\Pi_{j=0}^n\left((C_2 2^j)^{C_3 4^j}\right). \tag{30}$$

Since, by the invariance property, $N^{(n+1)}_{Q_L}(\frac{1}{2^{n+1}}) = N_{Q_L}(\frac{1}{2^{n+1}})$ then we infer from (30) that there exists C depending on the data to the equation such that

$$\frac{\log N^{(n_*+1)}_{Q_L}(\frac{1}{2^{(n_*+1)}})}{L} \le \log C \log\frac{1}{\varepsilon} + O(1). \tag{31}$$

Letting L go to infinity completes the proof of the Theorem. □

4. Lower bound for the ε-entropy.

In this section we establish the lower bound for the ε-entropy. This completes the proof of Theorem 1.3, i.e

Proposition 4.1. *There exists a constant C such that for ε small*

$$\frac{1}{C}\log(\frac{1}{\varepsilon}) \le H_\varepsilon.$$

Proof. We adapt here the method in [5] where the authors have proven that the unstable manifold at the fixed point 0 contains a set which has large enough ε-entropy. This method does not work straightforwardly (see the discussion in [14]), neither the method in [20], [21] to construct the unstable manifold at 0, and then we need to use another particular solution to CGL.

Consider L large enough and the restriction of the equation CGL to periodic functions on $Q_L = [-L, L]$. The global attractor of this periodic CGL equation is included into our global attractor. We now are in the periodic setting. The Laplacian operator has a discrete spectrum $\{\lambda_k = \frac{k^2\pi^2}{L^2}; k \in \mathbb{Z}\}$. There exists a particular family of time periodic solutions to CGL that read, for $\lambda_k < 1$,

$$u_k(t,x) = \sqrt{1-\lambda_k}\exp(-i\beta t)\exp(-i(\alpha-\beta)\lambda_k t)\exp(i\sqrt{\lambda_k}x). \tag{32}$$

We now fix the integer k such that

$$k\pi < L \leq (k+1)\pi. \tag{33}$$

The idea is to prove that the unstable manifold at the periodic solution u_k contains a set which has large enough ε-entropy. We set $\gamma = 1-\lambda_k$; then γ is small for large L, in fact

$$\gamma = 1-\lambda_k \leq \lambda_{k+1}-\lambda_k = \frac{(2k+1)\pi^2}{L^2} \preceq \frac{1}{L}. \tag{34}$$

To begin with, we linearize the equations around u_k. Setting $u(t,x) = u_k(t,x)(1 + w(t,x))$ we thus obtain

$$w_t - (1+i\alpha)(2i\sqrt{\lambda_k}w_x + w_{xx}) + (1-\lambda_k)(1+i\beta)(|1+w|^2-1)(1+w) = 0. \tag{35}$$

This equation reads in an abstract form as

$$w_t + Aw = F(w), \tag{36}$$

where the nonlinearity $F(w) = \gamma(1+i\beta)(w^2 + 2|w|^2 + |w|^2 w)$ is small and (up to a truncation for large $|w|$, which does not matter, since we look at w such that $|w| \simeq 0$) satisfies for $X = L^2(Q_L)$ or $L^\infty(\mathbb{R})$

$$||F(w)||_X \preceq \gamma ||w||_{L^\infty(\mathbb{R})}||w||_X. \tag{37}$$

We now study the unstable manifold of the associated linear equation. The two-dimensional space Π_m spanned by the plane waves $e_m(x) = \exp(i\sqrt{\lambda_m}x), e_{-m}(x) = \exp(-i\sqrt{\lambda_m}x)$ is invariant under the partial differential operator

$$Aw = -2(1+i\alpha)i\sqrt{\lambda_k}w_x - (1+i\alpha)w_{xx} + \gamma(1+i\beta)(2w+\overline{w}).$$

We now study A_m the matrix of A restricted to Π_m. This matrix A_m reads

$$\begin{pmatrix} (1+i\alpha)(2\sqrt{\lambda_k\lambda_m}+\lambda_m) + 2\gamma(1+i\beta) & \gamma(1+i\beta) \\ \gamma(1+i\beta) & (1+i\alpha)(-2\sqrt{\lambda_k\lambda_m}+\lambda_m) + 2\gamma(1+i\beta) \end{pmatrix} \tag{38}$$

Using this representation, we exhibit a large subspace into the unstable manifold for the linearized equation.

Proposition 4.2. *Assume L large enough (and then γ small enough). For m such that $\sqrt{\lambda_m} \in (\frac{2}{3}, \frac{4}{3})$ there exists an eigenvalue Λ_m^- for A_m such that $Re\Lambda_m^- \leq -\frac{2}{3}$. If E_m is the corresponding eigenvector and H the space spanned by these E_m, then the E_ms are a Riesz basis for H, i.e. there exist c a numerical constant, that is independent of L, such that*

$$\frac{1}{c}||\sum_m u_m E_m||_{L^2(Q_L)} \leq (\sum_m |u_m|^2)^{\frac{1}{2}} \leq c||\sum_m u_m E_m||_{L^2(Q_L)}.$$

Proof. The matrix has two eigenvalues

$$\Lambda_m^{\pm} = (1+i\alpha)\lambda_m + \gamma(1+i\beta) \pm (1+i\alpha)\sqrt{4\lambda_k\lambda_m + \gamma^2(\frac{1+i\beta}{1+i\alpha})^2}$$

where $\sqrt{z}$ is defined for any complex number $z = |z|e^{i\theta}$ such that $\theta \neq -\pi$ as $\sqrt{|z|}\exp(i\frac{\theta}{2})$. This is always possible choosing γ small enough.

We shall use now a perturbation method. Set $A_m(\gamma)$ for the matrix defined in (38) above. For $\gamma = 0$ we have $\mathrm{Re}\Lambda_m^- = \lambda_m + 2\sqrt{\lambda_m} = (\sqrt{\lambda_m} - 1)^2 - 1$. Then for $\sqrt{\lambda_m} \in [\frac{2}{3}, \frac{4}{3}]$ then $\mathrm{Re}\Lambda_m^- \geq -\frac{8}{9}$. On the other hand, introducing the matrix norm $||.||$ corresponding to the Euclidean structure on Π_m,

$$||A_m(\gamma) - A_m(0)|| \preceq \gamma. \tag{39}$$

We then apply the following classical Lemma (see [12] for instance)

Lemma 4.1. *Assume that (39) is valid. Then there exists a constant c such that for any eigenvalue Λ_γ of $A_m(\gamma)$ there exists an eigenvalue Λ of $A_m(0)$ such that $|\Lambda_\gamma - \Lambda| \leq c\gamma$.*

Proof. For μ that is not an eigenvalue of $A_m(0)$ we write

$$A_m(\gamma) - \mu Id = (A_m(0) - \mu Id)[Id + (A_m(0) - \mu Id)^{-1})(A_m(\gamma) - A_m(0))]. \tag{40}$$

Therefore as soon as $||A_m(\gamma) - A_m(0)|| < ||A_m(0) - \mu)^{-1}||^{-1} = \inf|\Lambda - \mu|$, the minimum computed on the eigenvalues of $A_m(0)$, then, thanks to Neumann lemma, $A_m(\gamma) - \mu Id$ is invertible. The result follows promptly. □

Using this perturbation Lemma we obtain that for γ small enough the result of the first assertion in the Proposition is granted. Let us observe that we also have $\mathrm{Re}\Lambda_m^- \geq -1 - c\gamma$ for any eigenvalue of A_m, using once again the perturbation result.

For Λ_m^-, the associated eigenvector is

$$E_m(x) = \exp(-i\sqrt{\lambda}_m x) - \frac{\gamma}{2\sqrt{\lambda_k\lambda_m} + \sqrt{4(1+i\alpha)^2\lambda_k\lambda_m + \gamma^2(1+i\beta)^2}}\exp(i\sqrt{\lambda}_m x).$$

Therefore $E_m(x) = \exp(-i\sqrt{\lambda}_m x) + O(\gamma)\exp(i\sqrt{\lambda}_m x)$ and

$$||\sum_m u_m E_m - \sum_m u_m e_{-m}||^2_{L^2(Q_L)} \preceq \gamma^2 \sum_m |u_m|^2, \tag{41}$$

and the proof of the Proposition is completed as soon as L large enough. □

Consider now H as in Proposition 4.2. We shall now flow a small ball in H into the unstable manifold for (35) at $w = 0$. Consider $\eta > 0$. Consider a point f in $H \cap B_{L^2(Q_L)}(0, \eta)$. We proceed as follows. For $T > 0$ we flow backward on the linear equation to compute $\exp(TA)f$; here we have a linear differential equation on a finite dimensional space and we can go backward in time. We now flow forward on the nonlinear equation to compute

$$f(t) = \exp((T-t)A)f + \int_0^t \exp((s-t)A)F(f(s))ds. \tag{42}$$

Then we consider the sequence $u_T = u(T)$. If this sequence converges, it converges to a point in the unstable manifold for (35) at $w = 0$. On the one hand, due to the upper bound on the real part of the spectrum of A/H, we have

$$||\exp((T-t)A)f||_{L^2(Q_L)} \leq \exp(-\frac{2}{3}(T-t))||f||_{L^2(Q_L)} \leq \eta\exp(-\frac{2}{3}(T-t)). \quad (43)$$

On the other hand, it is easy to prove that

$$||\exp((T-t)A)f||_{L^\infty(\mathbb{R})} \leq c\eta\sqrt{L}\exp(-\frac{2}{3}(T-t)); \quad (44)$$

in fact we just use that the E_m are bounded in L^∞ and the estimate

$$\sum_m |u_m| \preceq \sqrt{L}(\sum_m |u_m|^2)^{\frac{1}{2}}.$$

We now state and prove that $f(t)$ remains close to $\exp((T-t)A)f$ during this back and forth process, i.e. for $t \in [0,T]$,

Proposition 4.3. *There exists a numerical constant c such that for $t \in [0,T]$*

$$||f(t) - \exp((T-t)A)f||_{L^\infty(\mathbb{R})} \leq c\eta^2\exp(-\frac{4}{3}(T-t)).$$

Proof. We have the obvious estimate

$$||f(t) - \exp((T-t)A)f||_{L^\infty(\mathbb{R})} \leq \int_0^t ||e^{(s-t)A}||_{\mathcal{L}(L^\infty)}||F(f)||_{L^\infty(\mathbb{R})}ds. \quad (45)$$

On the one hand, we claim that

$$||e^{(s-t)A}||_{\mathcal{L}(L^\infty)} \leq c\exp((1+O(\gamma))(t-s)). \quad (46)$$

To check that (46) is valid we proceed as follows. For w that solves $w_t + Aw = 0$, we perform the change of variable $V(t,x) = \exp(-t)u_k(t,x)w(t,x)$; in fact we go back in the original variables. Hence V is solution to

$$V_t - (1+i\alpha)\Delta V + \gamma(1+i\beta)(3V + \frac{u_k^2}{|u_k|^2}\overline{V}) = 0, \quad (47)$$

or in Duhamel's form

$$V(t,x) = \exp((1+i\alpha)t\Delta)V_0 + \gamma\int_0^t \exp((1+i\alpha)(t-s)\Delta)((3V + \frac{u_k^2}{|u_k|^2}\overline{V}))ds. \quad (48)$$

Hence, since $||\exp((1+i\alpha)t\Delta)||_{\mathcal{L}(L^\infty(\mathbb{R}))} \leq 1$, then

$$||V(t)||_{L^\infty} \leq ||V(0)||_{L^\infty} + K\gamma\int_0^t ||V(s)||_{L^\infty}ds, \quad (49)$$

and by Gronwall lemma $||V(t)||_{L^\infty} \leq ||V(0)||_{L^\infty}\exp(O(\gamma)t)$. This leads to (46).

On the other hand we have the estimate

$$||F(f)||_{L^\infty(\mathbb{R})} \leq c\gamma||f(s)||^2_{L^\infty(\mathbb{R})} \leq 2c(||f(s) - e^{(T-s)A}f||^2_{L^\infty(\mathbb{R})} + ||e^{(T-s)A}f||^2_{L^\infty(\mathbb{R})}). \quad (50)$$

Introduce P the orthogonal projector onto H. We now use that f belongs to H and that the spectrum of PA is included in $[-1+O(\gamma), -\frac{2}{3}]$ to obtain

$$||e^{(T-s)A}f||^2_{L^\infty(\mathbb{R})} \preceq L||e^{(T-s)A}f||^2_{L^2(Q_L)} \leq L\eta^2 e^{-\frac{4}{3}(T-s)}. \quad (51)$$

Therefore $q(t) = ||f(t) - \exp(T-t)Af||_{L^\infty(\mathbb{R})}$ satisfies the differential inequality

$$q(t) \leq c\gamma L\eta^2 \int_0^t e^{t-s} e^{-\frac{4}{3}(T-s)} + c\gamma \int_0^t e^{t-s} q(s)^2 ds, \tag{52}$$

with the initial condition $q(0) = 0$. Therefore the estimate in Proposition 4.3 is valid for an interval $[0, T_{max})$. As long as $t < T_{max}$ we infer from (52)

$$q(t) \leq 3\eta^2 cL\gamma e^{-\frac{4}{3}(T-t)} + \tilde{c}\gamma\eta^4 e^{-\frac{8}{3}(T-t)}. \tag{53}$$

The right hand side of this inequality is bounded by $6\eta^2 cL\gamma e^{-\frac{4}{3}(T-t)}$. The result is then proved, using $\gamma L = O(1)$. □

We now complete the proof of the lower bound. The sequence $f(T)$ is bounded in $L^2(Q_L)$ and remains also bounded in smaller Sobolev spaces due to standard parabolic estimates. Up to a subsequence extraction, let us denote by $l(f) = \lim_{T\to+\infty} f(T)$. This limit belongs to the unstable manifold at u_k.

Due to Proposition 4.3, we have that for $||f||_{L^2(Q_L)} \leq \eta$ then

$$||f - l(f)||_{L^2(Q_L)} \leq ||f - l(f)||_{L^\infty} \leq c\eta^2. \tag{54}$$

Therefore we can conclude. Consider a small ball $\mathcal{B}_\eta$ in $H \cap L^2(Q_L)$ of radius η. Consider $\varepsilon = o(\eta)$. Due to Lemma 3.1, this small ball contains $O((\frac{\eta}{2\varepsilon})^L)$ points y_j such that $||y_j - y_i||_{L^2(Q_L)} \geq 2\varepsilon$. In fact $H \cap L^2(Q_L)$ is isometric to the euclidian $\mathbb{R}^{\text{dim}H}$ with $\text{dim}H \simeq L$. Due to (54)

$$||l(y_j) - l(y_i)||_{L^2(Q_L)} \geq 2\varepsilon - 2c\eta^2. \tag{55}$$

We chose now $\eta = \sqrt{\frac{\varepsilon}{2c}}$. Then the unstable manifold at $w = 0$ contains $O((\frac{1}{\varepsilon})^{\frac{L}{2}})$ points whose distance from one to one another is larger than ε. Therefore, using once again Lemma 3.1 and its proof, we obtain that we need $O((\frac{1}{\varepsilon})^{\frac{L}{2}})$ balls of radii ε to cover this set in $L^2(Q_L)$. Therefore

$$N_{Q_L}(\varepsilon) \geq C(\frac{1}{\varepsilon})^{\frac{L}{2}}, \tag{56}$$

and the result follows promptly. □

5. **Concluding remark.** In this article, we have addressed the issue of the existence of the entropy by unit length for the Ginzburg-Landau equation, following the guidelines in [5], but introducing a Hilbertian framework. The entropy depends on the metric considered on the infinite-dimensional phase space. To go further, we may address the following issue: as in [6], is the ε-entropy by unit length related to the topological entropy (by unit length) ? Does this topological quantity depend on the metric ? This issue will be addressed in a forthcoming work.

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