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AN ORIENTED MODEL FOR KHOVANOV HOMOLOGY

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ABSTRACT

We give an alternative presentation of Khovanov homology of links. The original construction rests on the Kauffman bracket model for the Jones polynomial, and the generators for the complex are enhanced Kauffman states. Here we use an oriented $sl(2)$ state model allowing a natural definition of the boundary operator as twisted action of morphisms belonging to a TQFT for trivalent graphs and surfaces. Functoriality in original Khovanov homology holds up to sign. Variants of Khovanov homology fixing functoriality were obtained by Clark–Morrison–Walker [7] and also by Caprau [6]. Our construction is similar to those variants. Here we work over integers, while the previous constructions were over gaussian integers.

Keywords: Link homology; categorification; Jones polynomial; Khovanov; TQFT; knot invariant.

Mathematics Subject Classification 2010: 57M25, 57M27

0. Introduction and Main Results

The Khovanov homology [10] produces bigraded homology groups which are link invariants and whose graded Euler characteristic is the unnormalized Jones polynomial. There are various nice expositions, see [1, 20] for the more basic ones. A functorial extension of a link homology assigns to a diagram of a link cobordism (a *movie*), a chain map whose action on homology is an invariant of the isotopy class of the cobordism with fixed ends. Functoriality for the original model based on Kauffman states holds up to sign [2, 8, 12]. This sign problem was solved by Clark–Morrison–Walker [7] and also by Caprau [6] by introducing a variant of Khovanov homology with coefficients in Gaussian integers. We give here a similar construction which works over integers. The model we use is based on the $N = 2$ case in the graphical calculus of Murakami–Ohtsuki–Yamada [17]. Our construction can be understood as $N = 2$ special case of $sl(N)$ link homology first constructed by Khovanov–Rozansky [13] with matrix factorizations and obtained via foams by McKaay–Stosic–Vaz [18]. As suggested by Manturov who considered Khovanov homology for virtual knots [15,16], it is very likely that our model extends to virtual knots.

To each oriented link diagram in the plane we associate a bigraded abelian group $K(D)$ with a self map δ and show the following Theorem 3.1.

Theorem 3.1. (a) *$(K(D), \delta)$ is a bigraded complex over $\mathbb{Z}$.*

(b) *If the diagrams D and D' are related by a Reidemeister move, then there exists a graded homotopy equivalence between the complexes $K(D)$ and $K(D')$.*

(c) *The graded Euler characteristic of $K(D)$ is equal to $q + q^{-1}$ times the Jones polynomial with change of variable $q = -t^{-\frac{1}{2}}$.*

We associate to a movie from a link diagram to another, a chain map, and show the following Theorem 5.1.

Theorem 5.1. *Let $C \subset [0,1] \times S^3$ be a smooth cobordism between the links L and L' represented by respective diagrams D and D'. The homology map* $\mathrm{Kh}(D) \to \mathrm{Kh}(D')$ *induced by a movie description of C only depends on the isotopy class of C rel. $L \times \{0\} \cup L' \times \{1\}$, and Kh extends to a functor on the embedded cobordism category.*

1. Trivalent TQFT

1.1. *Frobenius algebra*

Equivalences classes of TQFTs for oriented surfaces are in one to one correspondence with commutative Frobenius algebras up to isomorphism (see [9, 3.3.2 and historical remarks 3.3.4]. We consider here the Frobenius algebra $\mathbf{A} = \mathbb{Z}[X]/X^2 \approx H^*(\mathbb{C}P^1)$, and we denote by $V_{\mathbf{A}}$ the associated TQFT. The unit element in $\mathbf{A}$ is denoted by $\mathbf{1}$; the coalgebra structure (Δ, ϵ) on $\mathbf{A}$ is defined by

$$\epsilon(X) = 1, \quad \epsilon(\mathbf{1}) = 0;$$

$$\Delta(X) = X \otimes X, \quad \Delta(\mathbf{1}) = \mathbf{1} \otimes X + X \otimes \mathbf{1}.$$

The invariant of a closed surface is given below.

$$V_{\mathbf{A}}(S^1 \times S^1) = \mathrm{rank}(\mathbf{A}) = 2,$$

$$V_{\mathbf{A}}(\Sigma_g) = 0 \text{ for a closed surface } \Sigma_g \text{ with genus } g \neq 1.$$

The TQFT is extended to surfaces with points. The neighborhood of a point represents the element X in the algebra associated with the oriented circle $V_{\mathbf{A}}(S^1) = \mathbf{A}$. For a connected genus g closed surface with k points the invariant is zero except

- for $(g, k) = (1, 0)$ where the value is 2, and
- for $(g, k) = (0, 1)$ where the value is 1.

1.2. *The universal construction*

The universal construction for cobordism generated TQFT was pioneered in [5]. Let us sketch how we can reconstruct the above specific TQFT functor $V_{\mathbf{A}}$ from the invariant of closed surfaces with points denoted by I. The TQFT module of an oriented curve γ is generated over $\mathbb{Z}$ by surfaces with points whose boundary is identified with γ. Relations are given by the submodule

$$\mathcal{V}_0(\gamma) = \left\{ \sum_i \lambda_i M_i, \text{ for any cobordism } M = (M, \gamma, \emptyset), \ \sum_i I(M_i \cup_\gamma M) = 0 \right\}.$$

A key point in proving that the functor V_A defined this way is indeed a TQFT is the surgery formula in Fig. 1.

Here cobordisms are depicted from left to right. The graphical identity can be written

$$\mathrm{Id}_{\mathbf{A}} = \epsilon(X \times \cdot)\mathbf{1} + \epsilon(\cdot)X.$$

1.3. *Graded TQFT*

We define a grading on $\mathbf{A} = \mathbb{Z}[X]/X^2$, by $\deg(\mathbf{1}) = -1$, $\deg(X) = 1$. The q-dimension (or Poincaré polynomial) of the TQFT module associated with a k components curve is $(q + q^{-1})^k$. The TQFT functor is graded. For a cobordism Σ between γ and γ', the linear map

$$V_{\mathbf{A}}(\Sigma) : V_{\mathbf{A}}(\gamma) \to V_{\mathbf{A}}(\gamma')$$

has degree $-\chi(\Sigma) + 2\sharp\text{pts}$. Here $\chi(\Sigma)$ is the Euler characteristic, and $\sharp$pts is the number of points.

1.4. *Trivalent category*

We will extend the TQFT over the cobordism category whose objects are trivalent graphs and whose morphisms are trivalent surfaces. Here a *trivalent graph* is an oriented graph with edges labeled with 1 or 2, and 3-valent vertices where the flow condition is respected. For each trivalent vertex, an order on the 2 (germs of) edges labeled with 1 is fixed. For a planar graph we use plane orientation and fix the order according to the rule depicted in Fig. 2; the labels of the edges are obviously encoded in the arrows.

A closed trivalent surface is a 2-complex whose regular faces are oriented and labeled with 1 or 2. The singular locus is a curve called the binding; each component

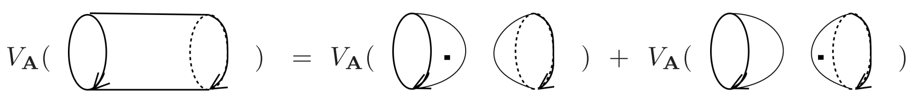


Fig. 1. Surgery formula.

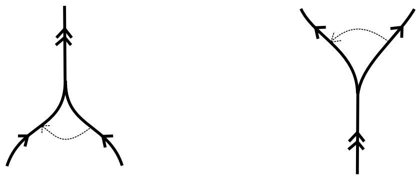


Fig. 2. Trivalent vertices.

of the binding has a neighborhood which is a triod times S^1, i.e. there are two 1-labeled pages inducing the same orientation and one 2-labeled page inducing the opposite orientation. For each component of the binding, an order on the two 1-labeled pages is fixed. A 1-labeled face may have points on it.

Cobordisms are obtained by cutting in a generic way. The boundary of a cobordism is a trivalent graph, and a component of the binding may be a triod times an interval. They are considered up to the usual equivalence of oriented homeomorphism rel. boundary.

1.5. *TQFT on the trivalent category*

The following general procedure constructs a functor on the trivalent cobordism category. We first define an invariant of closed trivalent surfaces, and extend it into a functor on the trivalent category via the universal construction introduced in [5] and sketched above (1.2) for surfaces. For the specific example used in the present paper, we will get a TQFT functor on the full subcategory whose objects are planar trivalent graphs.

Suppose that we are given Frobenius algebras A, B and C over a ring $\mathbf{k}$, with corresponding TQFT functors denoted by V_A, V_B and V_C. Let Σ be a closed trivalent surface, and $\dot{\Sigma} = \Sigma_1 \amalg \Sigma_2$ be the surface cut along the binding, decomposed according to the label of the faces. Let m be the number of components of the binding of Σ. The boundary of Σ_1 has $2m$ oriented components C_i^+ and C_i^-, $1 \leq i \leq m$, and the boundary of Σ_2 has m components C_i^2. Here the $\pm$ is fixed with respect to the ordering of the 1-labeled pages, i.e. C_i^+ is the boundary of the first page. The TQFT functors V_A and V_B associate to Σ_1 and Σ_2 vectors

$$V_A(\Sigma_1) \in \bigotimes_{i=1}^{m} (V_A(C_i^+) \otimes V_A(C_i^-)) \cong (A \otimes A)^{\otimes m},$$

$$V_B(\Sigma_2) \in \bigotimes_{i=1}^{m} V_B(C_i^2) \cong B^{\otimes m}.$$

Now suppose that we are given maps $f = A \otimes A \to C$, $g : B \to C$, then we define the invariant $\mathbf{V}(\Sigma)$ by the formula

$$\mathbf{V}(\Sigma) = (\epsilon_C)^{\otimes m}(f^{\otimes m}(V_A(\Sigma_1)) \times g^{\otimes m}(V_B(\Sigma_2)) \in \mathbf{k}^{\otimes m} = \mathbf{k}.$$

Here $\epsilon_C : C \to \mathbf{k}$ is the trace on the Frobenius algebra C; the product is computed in $C^{\otimes m}$.

From now on, we use the Frobenius algebras over $\mathbb{Z}$: $\mathbf{A} = \mathbb{Z}[X]/X^2 \approx H^*(\mathbb{C}P^1)$, $C = \mathbf{A}$, and $B = \mathbb{Z}$ with non standard trace $\epsilon_B(n) = -n$. We quote that the subtility which will give strict functoriality is contained in this non standard trace. If we chose the standard trace, the construction reproduces original Khovanov homology. The structural map f is defined by $f(x \otimes y) = x\overline{y}$, where $\overline{a + bX} = a - bX$ $(a, b \in \mathbb{Z})$, and $g : B = \mathbb{Z} \to C = \mathbf{A}$ is the unit map.

Example 1.1. Let us consider the trivalent surface which is a sphere together with a 2-labeled meridional disk, and whose 1-labeled half-spheres are ordered north-south. The associated value

- is 0 if there is no point,
- is 1 if there is one point which is on north half-sphere,
- is -1 if there is one point which is on south half-sphere,
- is 0 if there is more than one point.

The universal construction extends the invariant $\mathbf{V}$ to a functor on the trivalent cobordism category. The following proposition shows that the functor $\mathbf{V}$ is an extension of the TQFT functor V_A.

Proposition 1.2. (a) *We have a natural transformation from V_A to $\mathbf{V}$, i.e. for an oriented curve γ, we have a TQFT module $V_{\mathbf{A}}(\gamma)$, a module $\mathbf{V}(\gamma)$, and a natural map*

$$i_\gamma : V_{\mathbf{A}}(\gamma) \to \mathbf{V}(\gamma).$$

Here we label all the components of γ with 1, and consider γ as an object in the trivalent category.

(b) *For any curve γ, the natural map i_γ is an isomorphism.*

Proof. The surgery formula in Fig. 1 holds for surgery on a 1-labeled face of a trivalent surface. Using this formula along each component of the curve γ, we see that any trivalent surface with boundary γ representing a generator of $\mathbf{V}(\gamma)$ can be written as a linear combination of disks (may be with points), and that any linear combination representing a relation in $V_A(\gamma)$ also represents a relation in $\mathbf{V}(\gamma)$. This proves existence and surjectivity of i_γ. Injectivity of i_γ and naturality follows from the definitions in the universal construction. □

The extended functor $\mathbf{V}$ is still graded. The formula for a cobordism Σ is

$$\deg(\Sigma) = -\chi_1(\Sigma) + 2\sharp\text{pts}.$$

Here $\chi_1(\Sigma)$ is the Euler characteristic of the 1-labeled subsurface, e.g. the saddle with 2-labeled membrane in Fig. 3 has grading 1. Here 1-labeled faces are depicted in light grey and the 2-labeled half-disc is black.

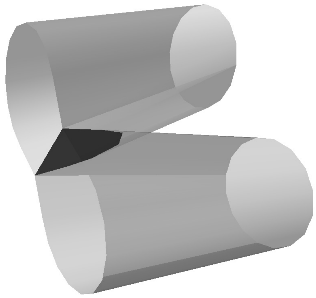


Fig. 3. Trivalent surface with grading 1.

The lemma below gives some examples of computation with the extended functor $\mathbf{V}$. These results will be useful in the subsequent categorification procedure. In the pictures, the order on the germs of 1-labeled edges is fixed by the following plane convention (Fig. 2): the first 1-labeled edge is on the right of the oriented 2-labeled adjacent edge. Proofs are left as exercise.

Lemma 1.3. (a) *If Σ' is obtained from Σ by moving a point across a component of the binding then* $\mathbf{V}(\Sigma') = -\mathbf{V}(\Sigma)$.

(b) *The bubble relations in Fig. 4 hold.*

(c) *The band moves relations in Fig. 5 hold.*

(d) *The tube relations in Fig. 6 hold* (*the sign depends on the order of the* 1*-labeled pages at each binding*).

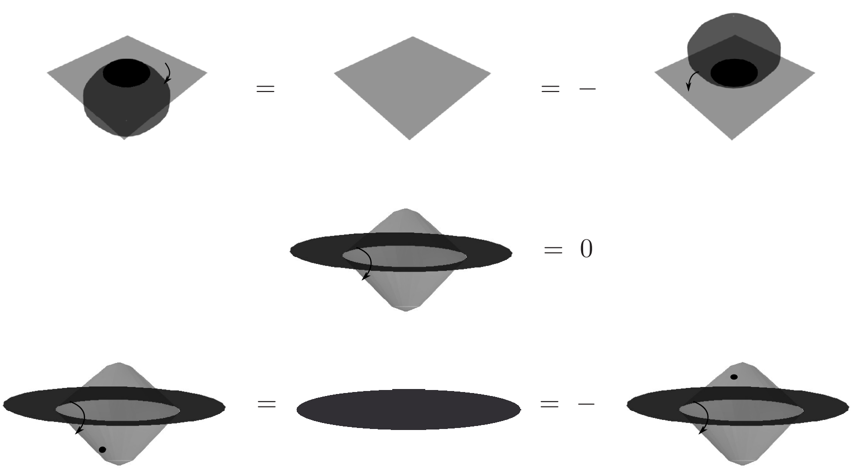


Fig. 4. Bubble relations.

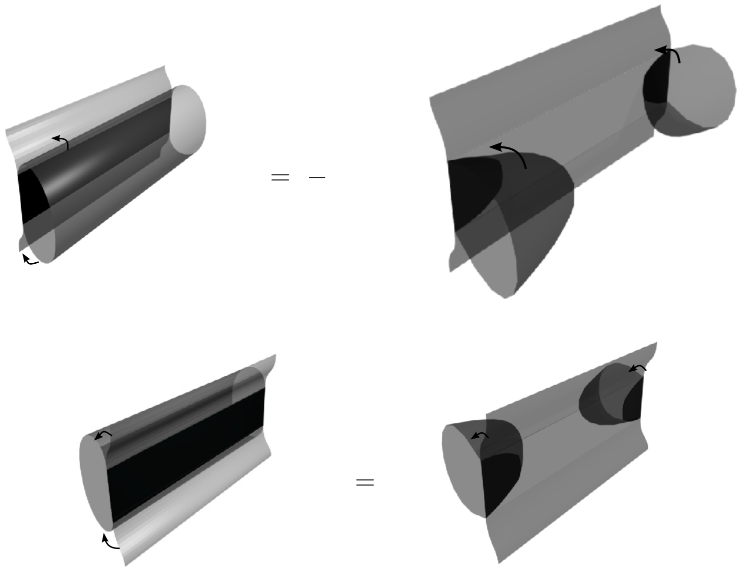


Fig. 5. Band relations.

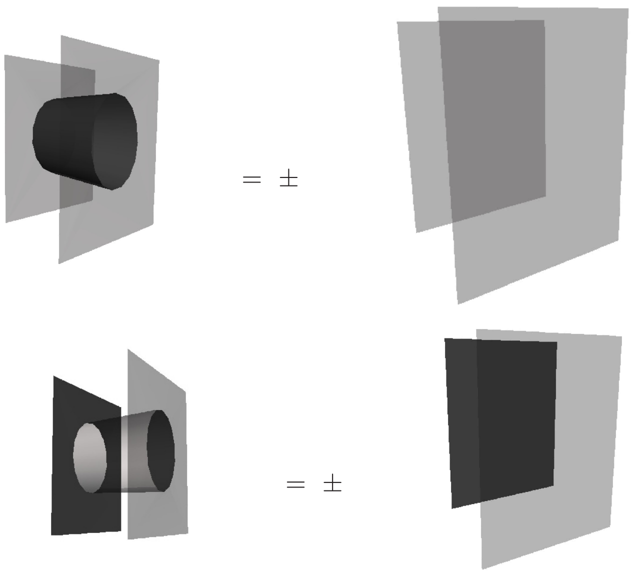


Fig. 6. Tube relations.

In the pictures, the 2-labeled faces are depicted in black, the 1-labeled faces are depicted in grey. The small arc indicates the order around the binding.

2. Categorification of the $sl(2)$ Invariant of Planar Graphs

We consider here trivalent planar graphs whose edges are smooth; each edge has a label equal to 1 or 2. In each trivalent vertex, the flow is conserved, and the tangent vectors are coherent. Loops with label 1 are accepted. The admissible vertices are depicted in Fig. 2. In the representation theoretic setting, 1-labeled edges correspond to the standard representation of $sl(2)$, and 2-labeled edges correspond to its determinant (isomorphic to the trivial representation).

An enhancement ϵ for such a graph is a map from the set of 1-labeled edges to $\{-1, 1\}$ required to have distinct values for the two edges adjacent to a trivalent vertex. To each trivalent vertex v we associate a weight $\mathcal{W}(v) = q^{\pm\frac{1}{2}}$. Here q is an indeterminate, and the sign is given by the state of the right handed edge. The $sl(2)$ invariant of such a graph G is given by

$$\langle G \rangle = \prod_{\text{vertices } v} \mathcal{W}(v) q^{\sum_a \epsilon(a)\mathrm{rot}(a)}.$$

The sum is over all 1-labeled edges a, and $\mathrm{rot}(a)$ is the variation of the tangent vector along the edge, normalized so that it gives the Whitney degree (signed number of rotation) for a closed curve.

The invariant $\langle G \rangle$ is easily seen to be equal to $(q + q^{-1})^{\sharp G_1}$ where $\sharp G_1$ is the number of components of the curve composed with the 1-labeled edges. Its interest is that it allows to give a state model for the Jones polynomial similar to the Kauffman bracket state model, but taking into account the orientation.

We associate to such a graph the graded module $\mathbf{V}(G) = \oplus_k \mathbf{V}_k(G)$. For any graph G, the module $\mathbf{V}(G)$ admits a finite set of generators which can be obtained by first pairing the trivalent vertices with singular arcs and then gluing discs, may be with points on the 1-labeled ones. The module itself can then be computed using the pairing.

Exercise 2.1. Let G_1, G_2, G_3, G_4 be the graphs depicted in Fig. 7.

(a) Show that

$$\mathbf{V}(G_1) \approx \mathbf{A}, \quad \mathbf{V}(G_2) \approx \mathbf{A}^{\otimes 2}.$$

(b) (**) Show that

$$\mathbf{V}(G_3) \approx \mathbf{A}^{\otimes 2}, \quad \mathbf{V}(G_4) \approx \mathbf{A}.$$

(Use the grading to have an upper bound on the number of points.)

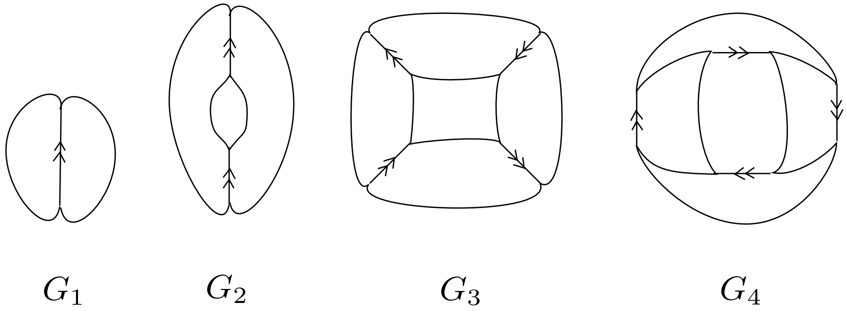


Fig. 7.

Proposition 2.2. *For any planar trivalent graph G, the module $\mathbf{V}(G)$ is a free abelian group whose q-dimension is equal to the invariant $\langle G \rangle$:*

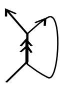
$$\langle G \rangle = \sum_k q^k \operatorname{rank}(\mathbf{V}_k(G)).$$

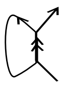
Remark 2.3. We understand this result as a categorification of the invariant $\langle G \rangle$. Indeed, the functor $\mathbf{V}$ associates to a graph G a graded abelian group which can be interpreted as (co)homology concentrated in (co)homological degree zero. The purpose of the next section is to extend this categorification to link diagrams.

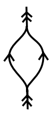
Remark 2.4. It follows that $\mathbf{V}_A$ is multiplicative for disjoint union of planar graphs. Hence $\mathbf{V}$ is a TQFT functor on the full trivalent subcategory whose objects are planar trivalent graphs.

Proof. We proceed by induction on the number of 2-labeled edges. If this number is 0, we get a curve γ. From Proposition 1.2, the module $\mathbf{V}(\gamma)$ is $\mathbf{A}^{\otimes \sharp\gamma}$ whose graded dimension is $(q+q^{-1})^{\sharp\gamma}$.

Consider a connected component F of the graph G which is not a circle. Each face of F has an even number of edges (F is bipartite). By an Euler characteristic argument the graph F has at least one face which is either a bigon or a square. Lemmas 2.5 and 2.7 below shows that the computation reduces to graphs with less 2-labeled edges. □

Lemma 2.5 (Bigons). (a) $\mathbf{V}($  $) \simeq \mathbf{V}($  $)$.

(b) $\mathbf{V}($  $) \simeq \mathbf{V}($  $)$.

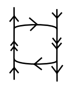
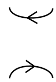
(c) $\mathbf{V}($  $) \simeq \mathbf{V}($  $)\{-1\} \oplus \mathbf{V}($  $)\{1\}$.

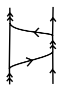
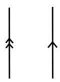
Here the bracket in right hand side of (c) indicates a shift in the grading.

Proof. (a) and (b) are deduced from the band relations in Lemma 1.3(c). Indeed the cobordisms in the right-hand side of these relations can be decomposed by cutting in the middle. The induced TQFT maps give the needed isomorphisms.

Lemma 2.6 below, whose proof is left to the reader, decomposes identity of the module on the left-hand side of (c) into two orthogonal idempotents whence the direct sum decomposition. Note the shift given by the degree of the cobordisms inducing the projection on each summand. □

Lemma 2.6. *The relations in Fig. 8 hold.*

Lemma 2.7 (Squares). (a) $\mathbf{V}($  $) \simeq \mathbf{V}($  $)$.

(b) $\mathbf{V}($  $) \simeq \mathbf{V}($  $)$.

Proof. Isomorphisms in (a) are depicted in Fig. 9. Both compositions give identity up to sign. One can be seen using relations in Lemma 1.3. The second one is an identity in $\mathbf{V}(G_3)$ where G_3 is the graph described in Fig. 7, which can be checked by pairing with degree -2 generators of $\mathbf{V}(G_3)$. Isomorphisms in (b) are described in Fig. 10. □

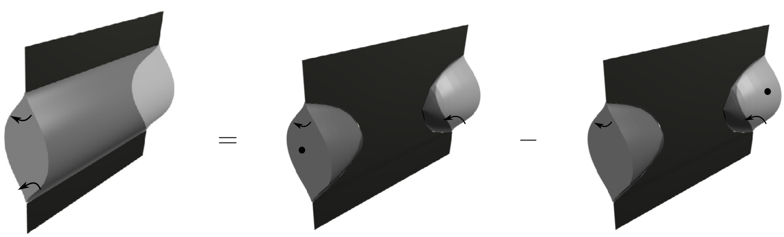


Fig. 8. Bigon relation.

Fig. 9. Isomorphisms in Lemma 2.7(a).

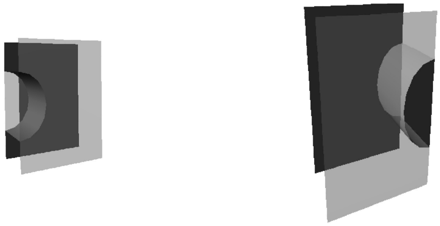
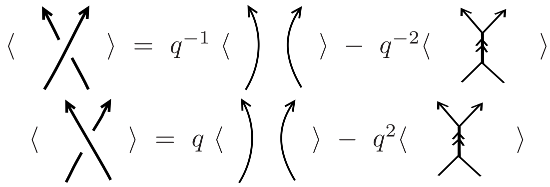


Fig. 10. Isomorphisms in Lemma 2.7(b).

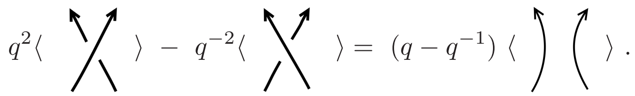
3. Khovanov Homology

3.1. *Jones polynomial via planar graphs*

The formulas below extend the preceeding $sl(2)$ invariant of planar trivalent graphs to an invariant of link diagrams. This is $N = 2$ case in [17].



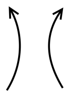
The normalization for the empty link is 1, and we have the following skein relation

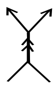


(3.1)

Up to normalization, we recognize the Jones polynomial with the change of variable $q = -t^{-\frac{1}{2}}$. A global state sum formula for a link represented by a diagram D is given below. Note that it is quite easy to show that this formula is invariant under Reidemeister moves; this is a slight variant of the Kauffman bracket construction.

We give a global state sum formula for a diagram D. A state s of D associates to a positive (respectively, negative) crossing either 0 or 1 (respectively, -1 or 0). For a state s, D_s is the planar trivalent graph, defined by the rule:

if $s(c) = 0$, then c is replaced by 

if $|s(c)| = 1$, then c is replaced by  .

One has

$$\langle D\rangle = \sum_s q^{-(w(D)+s(D))}\langle D_s\rangle. \tag{3.2}$$

Here $w(D) = \sum_c \operatorname{sign}(c)$, and $s(D) = \sum_c s(c)$.

3.2. *Khovanov complex*

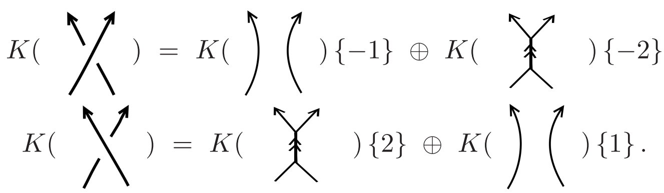
We consider a link diagram D. For a state s, we define the trivalent graph D_s according to the local rules described just above. We use the notation d_s for $\sum |s(c)|$, and Δ_s for the free abelian group generated by crossings c with $|s(c)| = 1$. The Khovanov complex is a bigraded abelian group $K(D)$ defined below, together with a convenient boundary operator.

$$K(D) = \bigoplus_s V(D_s) \left\{ -\sum_c (\mathrm{sign}(c) + s(c)) \right\} \otimes \wedge^{d_s} \Delta_s. \tag{3.3}$$

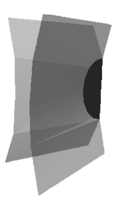
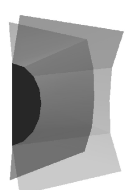
The cohomological degree $s(D) = \sum_c s(c)$ will be called the height, and the graded degree, equal to the one in the TQFT functor $\mathbf{V}$, up to a shift, will simply be called the degree. The shift from the TQFT degree is prescribed by the integer between braces in such a way that

$$\text{q-dim}(G\{i\}) = q^i \ \text{q-dim}(G).$$

It is convenient to give a local description of the complex. Here we implicitly extend the definition of K to trivalent diagrams, where only 1-labeled edges are allowed for crossings.



The boundary operator between summands indexed by states s and s' is zero unless s and s' are different only in one crossing c, where $s'(c) = s(c) + 1$. For a positive crossing (respectively, a negative crossing) it is then defined using the TQFT map associated with the cobordism Σ, (respectively, Σ') which are identity outside a neighbourhood of the crossing, and are given by a saddle with 2-labeled membrane, around the crossing c with $s(c) = 0$, $s'(c) = 1$ (respectively, $s(c) = -1$, $s'(c) = 0$).

Σ :  Σ' : 

For a positive crossing c:

$$\delta = \mathbf{V}(\Sigma) \otimes (\bullet \wedge c) : \mathbf{V}(D_s) \otimes \wedge^{d_s} \Delta_s \to \mathbf{V}(D_{s'}) \otimes \wedge^{d_{s'}} \Delta_{s'}.$$

For a negative crossing c,

$$\delta = \mathbf{V}(\Sigma')\otimes < \bullet, c >: \mathbf{V}(D_s) \otimes \wedge^{d_s}\Delta_s \to \mathbf{V}(D_{s'}) \otimes \wedge^{d_{s'}}\Delta_{s'}.$$

Here $< \bullet, c >$ is (the antisymmetrization of) the contraction (using the standard scalar product we understand c as a form).

Theorem 3.1. (a) $(K(D), \delta)$ *is a graded complex.*
(b) *If the diagrams D and D' are related by a Reidemeister move, then there exists a graded homotopy equivalence between the complexes $K(D)$ and $K(D')$.*
(c) *The graded Euler characteristic of $K(D)$ is equal to the quantum invariant $\langle D\rangle$, i.e to $q+q^{-1}$ times the Jones polynomial with change of variable $q = -t^{-\frac{1}{2}}$.*

We will use the notation $\mathrm{Kh}(D)$ for the homology of the complex $K(D)$. This theorem says that the isomorphism class of the graded group $\mathrm{Kh}(D)$ is an invariant of the isotopy class of the corresponding link.

Proof. We first prove (a). The map ∂ increases the height by one. The elementary cobordism given by a saddle has Euler characteristic -1. The corresponding TQFT map has degree $+1$, and the map ∂ on the shifted TQFT groups has degree 0. We want now to compute $\partial \circ \partial$. The possibly non trivial contributions come from squares corresponding to states s and s'' identical on all crossings except c_1 and c_2, where $s''(c_1) = s(c_1)+1$ and $s''(c_2) = s(c_2)+1$. We have two intermediate states s'_1 ($s'_1(c_1) = s(c_1)+1$ and $s'_1(c_2) = s(c_2)$) and s'_2 ($s'_2(c_1) = s(c_1)$ and $s'_2(c_2) = s(c_2)+1$), giving two contributions represented by the same cobordism Σ with two saddles (for the TQFT maps, squares commute). Each of them is twisted. We have to check that after twisting the two contributions vanish (squares anticommute) in all cases.

If c_1 and c_2 are both positive crossings, then we get

$$\mathbf{V}(\Sigma) \otimes (\bullet \wedge c_1 \wedge c_2 + \bullet \wedge c_2 \wedge c_1) = 0.$$

If c_1 and c_2 are both negative crossings, then we get

$$\mathbf{V}(\Sigma) \otimes (< \bullet, c_1 \wedge c_2 > + < \bullet, c_2 \wedge c_1 >) = 0.$$

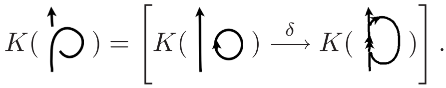
If c_1 is a positive crossing and c_2 is a negative crossing, then we get

$$\mathbf{V}(\Sigma) \otimes (< \bullet \wedge c_1, c_2 > + < \bullet \wedge c_1 > \wedge c_2) = 0.$$

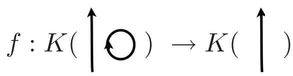
The graded Euler characteristic of the complex $K(D)$ satisfies the Jones skein relation (3.1), and is equal to $q+q^{-1}$ for the trivial diagram. Statement (c) follows. In the next subsections, we will construct homotopy equivalences for each Reidemeister move, and obtain (b). □

3.3. *Reidemeister move I*

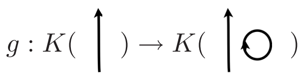
We first consider the case of a positive crossing.



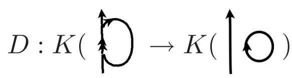
Recall that the map δ is equal to $\mathbf{V}(\Sigma_\delta) \otimes (\bullet \wedge c)$ where Σ_δ contains a saddle with membrane as depicted in Fig. 11.
Consider the followings maps.



is the TQFT map associated with the cobordism in Fig. 12.



is the sum of the TQFT maps associated with the cobordisms in Fig. 13.



is equal to $-\mathbf{V}(\Sigma_D) \otimes < \bullet, c >$ where Σ_D is depicted in Fig. 14.

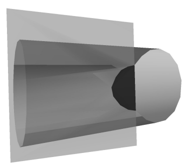


Fig. 11.

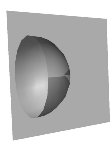


Fig. 12.

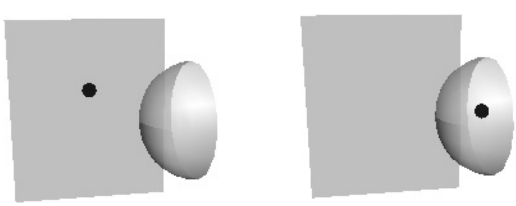


Fig. 13.

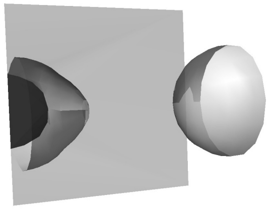


Fig. 14.

We have that

$$\delta \circ g = 0.$$

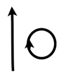
Hence f and g define chain maps. We have that

$$f \circ g = \mathrm{Id},$$

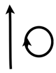
$$\mathrm{Id} - g \circ f = D \circ \delta \quad \text{and} \quad \delta \circ D = \mathrm{Id}.$$

Hence we have that D is an homotopy between Id and $g \circ f$.

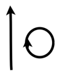
We consider now the case of a negative crossing.

$K($  $) = \Big[K($  $) \xrightarrow{\delta} K($  $S) \Big]$

Here the map δ is equal to $\mathbf{V}(\Sigma_\delta) \otimes < \bullet, c >$ where Σ_δ is a saddle. Consider the followings maps.

$f : K($  $) \to K($  $)$

is the TQFT map associated with the cobordism in Fig. 12.

$g : K($  $) \to K($  $)$

is the sum of the TQFT maps associated with the cobordisms in Fig. 13.

$D : K($  $\to K($  $)$

is equal to $\mathbf{V}(\Sigma_D) \otimes < \bullet, c >$ where Σ_D is depicted in Fig. 15.

We have that

$$f \circ \delta = 0.$$

Hence, f and g define chain maps. We have that

$$f \circ g = \mathrm{Id},$$

$$\mathrm{Id} - g \circ f = \delta \circ D \quad \text{and} \quad D \circ \delta = \mathrm{Id}.$$

Hence, we have that D is a homotopy between Id and $g \circ f$.

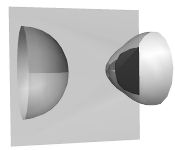
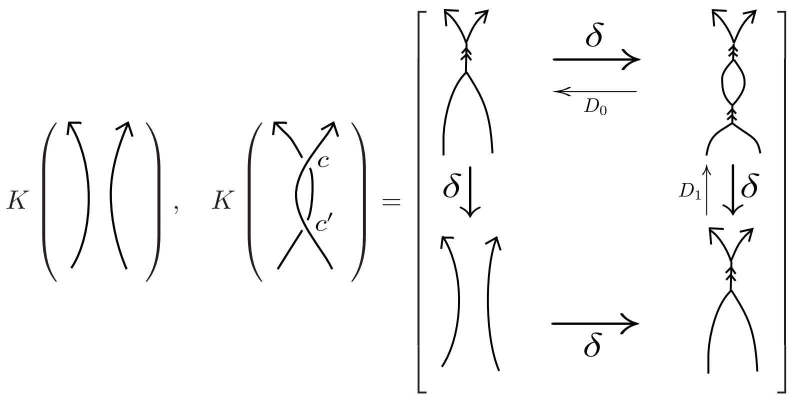


Fig. 15.

3.4. *Reidemeister II_+*

The complexes we want to consider are described in the diagram below.



Inverse homotopy equivalences are given by the maps f and g defined below

$$f = \mathbf{1}_{)(} \oplus \mathbf{V}(Z) \otimes (\bullet \wedge c \wedge c'),$$

$$g = \mathbf{1}_{)(} \oplus \mathbf{V}(Z') \otimes < \bullet, c \wedge c' > .$$

Here Z and Z' are the trivalent surfaces depicted in Fig. 16. One can check that $\delta \circ f = 0$ and $g \circ \delta = 0$, so that f and g define chain maps. We have $f \circ g = \mathrm{Id}$,

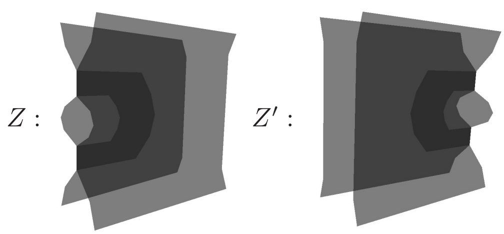


Fig. 16.

moreover there exists D_0, D_1 as depicted in the diagram above such that

$$\delta \circ D_1 = \mathrm{Id}, \quad D_0 \circ \delta = \mathrm{Id}, \quad g \circ f + D_1 \circ \delta + \delta \circ D_0 = \mathrm{Id}.$$

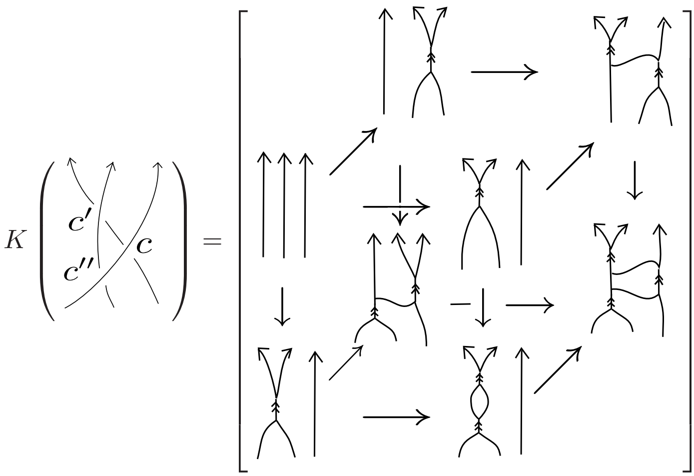
Exercise 3.2. Find D_0, D_1 as expected (hint: use the bigon relation 2.6).

3.5. *Reidemeister II*$_-$

Homotopy equivalences for negative Reidemeister move are defined in a similar way. The homotopy equivalence checking rests essentially on Lemma 2.7(a).

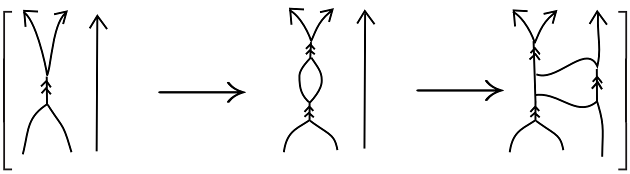
3.6. *Reidemeister III*

We have to consider the complex decomposed as a cube as described below, and the symmetric one.



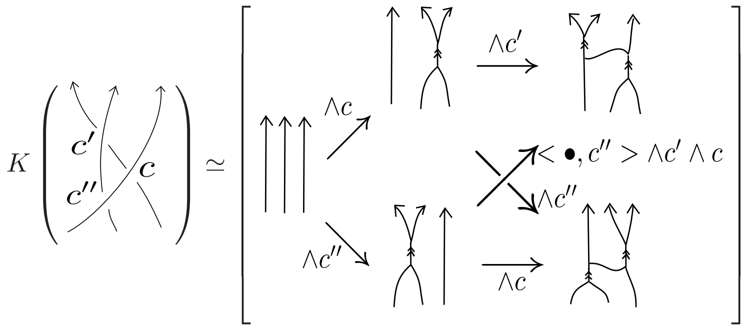
We can find an acyclic subcomplex.

Lemma 3.3. *The subcomplex described below is acyclic.*

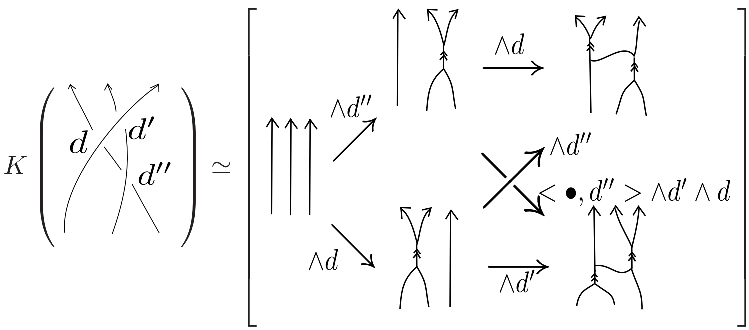


Proof. The proof rests essentially on Lemma 2.7(b). □

We then apply a Gauss reduction (see e.g. [3, Lemma 4.2]) and obtain a homotopy equivalence with a smaller comlex. Note that the above acyclic subcomplex is not a direct summand so that we have to carefully recalculate the boundary map including the action on the twisting determinant. The result is described below Here the boundary maps are given by a saddle with a 2 labeled membrane as before, tensored with a map on top exterior powers corresponding to the indication near the arrows.



We have



An isomorphism between the two complexes is obtained by using the idendity map on the TQFT modules, and $c \mapsto d'', c' \mapsto d', c'' \mapsto d$ on the twisting determinants.

4. Lee–Rasmussen Spectral Sequence

Following Lee and Rasmussen [14, 19], we consider now the Frobenius algebra $\mathbf{A}' = \mathbb{Z}[X]/X^2 - 1$, and the associated TQFT functor $\mathbf{V}'$. The preceeding construction applies as well, excepted that here the TQFT functor $\mathbf{V}'$ is no more graded, but filtered. A generator of a TQFT module $\mathbf{V}'(\gamma)$ is given the same degree as before, and $F^j(\mathbf{V}'(\gamma))$ is spanned by generators with degree less than or equal to j. Observe that a cobordism Σ induces a filtered map, with degree equal to $-\chi(\Sigma) + 2(\sharp\text{points})$.

We consider the filtered abelian group $K'(D)$ defined below, with the same notation as before.

$$K'(D) = \bigoplus_s \mathbf{V}'(D_s)\{-\sum_c(\text{sign}(c) + s(c))\} \otimes \wedge^{d_s}\Delta_s. \tag{4.1}$$

The boundary operator ∂' is still defined by the TQFT map associated with a saddle tensored with the natural map on the top exterior power. We denote by $\text{Kh}'(D)$ the homology of this complex.

Theorem 4.1. (a) $(K'(D), \partial')$ *is a filtered chain complex.*

(b) *If the diagrams D and D' are related by a Reidemeister move, then there exists a filtered homotopy equivalence between the complexes $K'(D)$ and $K'(D')$.*

(c) *There exists a spectral sequence whose second page is the homology of our complex K*

$$E_2^{i,j}(D) = \mathrm{Kh}^{i-j,j}(D),$$

which converges to $\mathrm{Kh}'^{*}(D)$*.*

Proof. (a) and (b) are proved as before. Statement (c) follows from standard facts with filtered chain complexes. □

The theorem shows that all the pages with index greater or equal to 2 are invariants of the link. For Khovanov original homology, it was proved by Lee [14] that the limit depends only on the number of components. Rasmussen [19] was able to extract a lower bound for the slice genus and to use it to give a combinatorial proof of Milnor conjecture on the slice genus of torus knots.

We will compute our oriented version of Lee–Rasmussen homology over $\Lambda = \mathbb{Z}[\frac{1}{2}]$ using the Karoubi completion method of Bar-Natan and Morrison [4]. We denote by $\mathrm{Kh}'(D, \Lambda)$ the homology of $K'(D) \otimes \Lambda$.

Theorem 4.2. *For a link diagram D with m components,* $\mathrm{Kh}'(D, \Lambda)$ *is a free Λ-module of rank 2^m, with a canonical basis indexed by maps $\epsilon : \pi_0(D) \to \{\pm 1\}$.*

Proof. The algebra $\mathbf{A}'$ contains the minimal idempotents

$$\pi_{\pm} = \frac{\mathbf{1} \pm X}{2}.$$

Using these idempotents we extend the TQFT functor $\mathbf{V}'$ to an extended trivalent category where 1-labeled edges or faces may be colored with $\pi_{\pm}$. If 1-labeled edges in a trivalent graph γ are colored with a sequence of signs denoted by ϵ, then we obtain an object $\gamma(\epsilon)$ whose associated module is the image of the obvious projector $\pi_\epsilon \in \mathbf{V}'(\gamma)$, associated with ϵ. The relation in Lemma 1.3(a) still holds for the functor $\mathbf{V}'$. We deduce that the module $\mathbf{V}'(\gamma)$ is zero if signs agree on the two 1-labeled edges adjacent to a vertex. For an *alternating* sign assignment ϵ, the module $\mathbf{V}'(\gamma(\epsilon))$ has rank one with a basis represented by a trivalent surface with only discs (without points) as 2-cells. The complex $K'(D)$ which was a sum indexed by states, is now decomposed into a sum indexed by colored states (states with coloring of all arcs). The boundary map can be computed locally. The TQFT map associated to a saddle is zero unless all colors coincide on the arcs belonging to the same 1-labeled component. In the remaining case, this TQFT map is an isomorphism. We obtain a deformation retract on a subcomplex $K''(D)$ where the boudary map is zero. Locally, i.e. for a crossing, the subcomplex $K''(D)$ is described in Fig. 17. For a generator, the signs associated with arcs belonging to the same

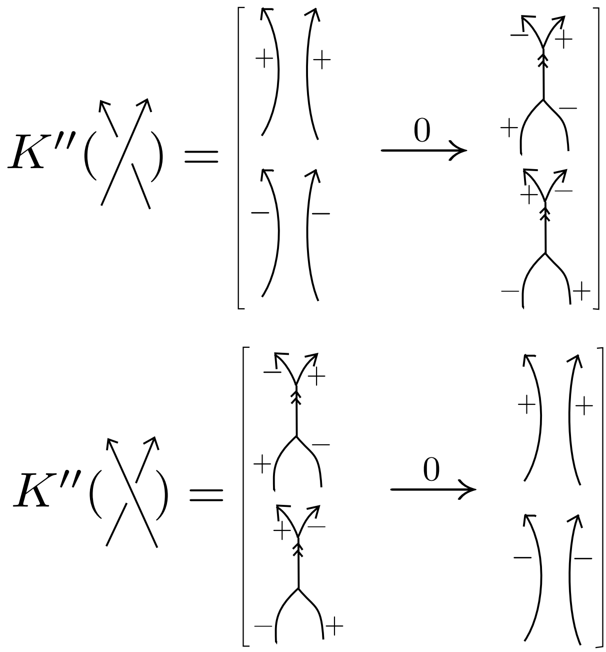


Fig. 17. The subcomplex with vanishing differential.

component of the represented link are the same. Moreover, for an assignment of signs on the components, the state of each crossing is determined so there is a unique corresponding generator. □

5. Functoriality

Extension of Khovanov homology to link cobordisms and functoriality up to sign was conjectured by Khovanov and established by Jacobson [8], Khovanov [12] and Bar-Natan [2]. The sign ambiguity was carried over by Clark–Morrison–Walker [7] and also by Caprau [6]. In this section we show that our construction has a strictly functorial extension to link cobordism.

A movie description of a cobordism is a generic projection in $[0,1]\times\mathbb{R}^2$ of a smooth surface in $[0,1]\times\mathbb{R}^3$. Generically, a movie decomposes into elementary ones which either describe a Reidemeister move or glue a handle to the surface. To each Reidemeister type movie we associate the corresponding homotopy equivalence, and to each handle addition we associate the corresponding TQFT map. We then compose these elementary chain maps.

Theorem 5.1. *Let $C\subset[0,1]\times S^3$ be a smooth cobordism between the links L and L' represented by respective diagrams D and D'. The homology map* $\mathrm{Kh}(D)\to\mathrm{Kh}(D')$ *induced by a movie description of C only depends on the isotopy class of*

C rel. $L \times \{0\} \cup L' \times \{1\}$, and Kh *extends to a functor on the embedded cobordism category.*

Remark 5.2. Note that here we consider fixed links, and not links up to isotopy. It was observed by Jacobson that Khovanov homology does have monodromy.

Proof. The point is to check that for a list of movie moves the two induced homology maps coincide. We borrow from Bar Natan [2] the argument of simplicity showing projective functoriality, which works here as well. We get that for each movie move MM, the two maps on the filtered complexes $K'(.)$ are filtered homotopic up to a sign $\epsilon(\mathrm{MM})$. The corresponding two maps on the homologies Kh(.) are equal up to the same sign $\epsilon(\mathrm{MM})$. For each movie move MM, we have to show that the sign $\epsilon(\mathrm{MM})$ is 1.

In Theorem 4.2, we gave a basis for $\mathrm{Kh}'(L)$. It contains a distinguished element associated with the assignment of a positive sign to all components. This canonical element is respected by the map associated with a Reidemeister move II+ or III. In Reidemeister I, it is respected up to a coefficient $\frac{1}{2}$ and 2 respectively for maps f and g. We also get $\frac{1}{2}$ and 2 coefficient in RII- for a natural choice of homotopies. For a 1-handle the canonical element is respected, up to a coefficient 2 in the case where the number of components increases by one (because $\Delta(p+) = 2p_+ \otimes p_+$). For a 2-handle we get a coefficient $\frac{1}{2}$. For a 0-handle, the canonical basis element y_+ is sent to $y_+ \otimes 1$ (which is not the canonical one). For all the movie moves except those with a 0-handle, we will get a positive coefficient, and we are done. By a small computation we see that the three remaining moves MM11, MM12, MM14 in [2, Fig. 13] are also satisfied. □

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