Univ. Internationale de Rabat

Topological Robotics

Arm Robot Motion

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Summary

بِسمِ اللَّهِ الرَّحمَنِ الرَّحِيمِ

وَ عَلَى اللَّهِ فَليَتَوَكَّلِ المُتَوَكِّلُون

صَدَقَ اللَّهُ العَظِيم

1. Robotic
2. Robot Motion Planning
3. Topological Robotics
4. Topological Complexity
5. Motion in Spheres
6. Arm Robot Motion

Introduction

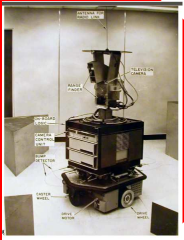
- The ultimate goal of robotics is creating of autonomous robots
- Such robots should be able to accept high-level descriptions of tasks and execute them without further human intervention. The input description specifies what should be done and the robot decides how to do it and performs the task.
- The idea of robots goes back to ancient times.
- The word robot was first used in 1921 by Karel Capek in his play “Possum’s Universal Robots” .
- The word robotics was coined by Isaac Asimov in 1940 in his book “I, robot.”

Robots History

no sensing necessary

exact models

Classical Robotics (mid-70’s)



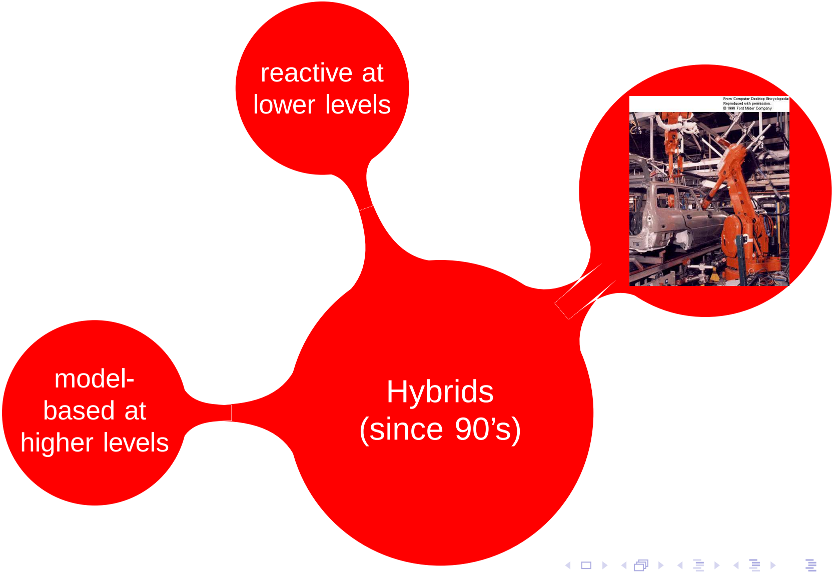
Robots History

relies heavily on good sensing

no models

Reactive Paradigm (mid-80's)

Robots History



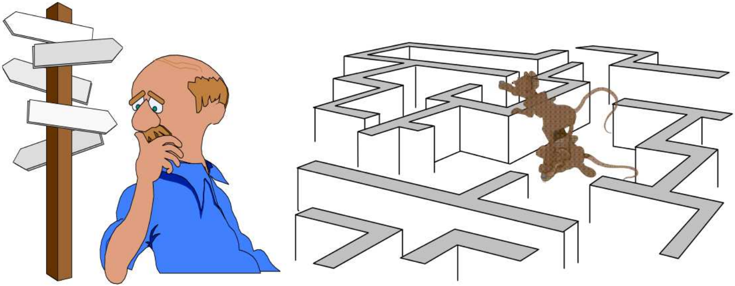
Robots History

inaccurate models, inaccurate sensors

seamless integration of models and sensing

Probabilistic Robotics (since mid-90's)

General Principle



General Principle

Imagine

You get into your advanced car and say “Go home!” and the car takes you home, automatically, obeying the trafic rules. Such a car must have a GPS (finding its current location) and a computer program suggesting a specific route from any initial state to any desired state.

Computer programs of this are based on motion planning algorithms.

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The founder

Jean Claude Latombe
Stanford Univ. USA

Edition Kluwer
(1991)

Mathematization

In general, a motion planning algorithm

is a function which assigns to any pair of states of the system (i.e., the initial state and the desired state) a continuous motion of the system starting at the initial state and ending at the desired state.

Definition

Let X be a topological-space, a Motion Planning Algorithm in X takes as input a pair $(A, B) \in X \times X$ and outputs a path $\gamma : [0; 1] \longrightarrow X$ from A to B There always exists such a path when X is path-connected.

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Motion Planning Algorithm

Consider the endpoint map

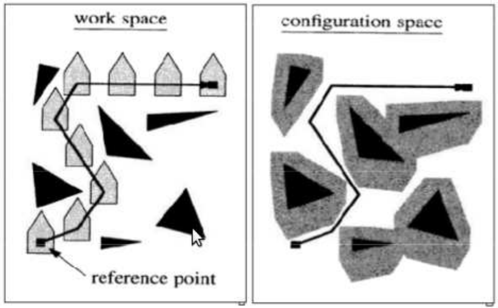
$$\begin{array}{rcl} \pi : \quad PX := X^{[0,1]} & \longrightarrow & X \times X \\ \gamma & \longmapsto & (\gamma(0), \gamma(1)) \end{array}$$

π is surjectiv when X is path-connected. Then an MPA in X is a section of this map, that is, a function $s : X \times X \longrightarrow PX$ such that $s \circ \pi = id$, i.e. $s.\pi(A, B) = (A, B)$

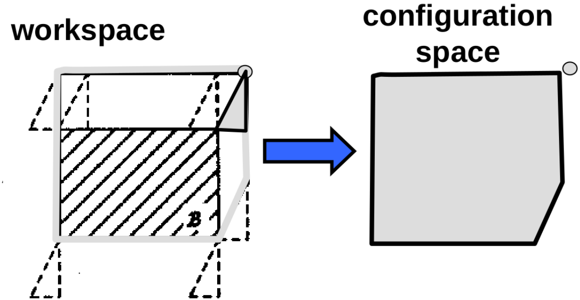
Robotics & Topology

What is common to robotics and topology ?

Topology enters robotics through the notion of configuration space. Any mechanical system R determines the variety of all its possible states X which is called the configuration space of R. Each point of X represents a state of the system.

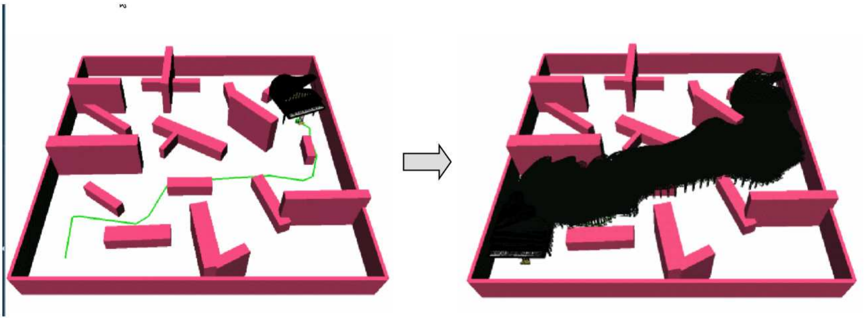


Robotics & Topology



Polygonal robot translating in 2D

Robotics & Topology



Piano movers' problem ; Schwartz and Sharir, 1983

Robotics & Topology

Equip the path space PX with compact-open topology,

then we can talk about nearby paths.

Robotics & Topology

What is common to robotics and topology ?

- Does there exist a continuous motion planning in X ?
- In other words does there exist a motion planning in X such that the section $s : X \times X \longrightarrow PX$ is continuous ?
- Equivalently, it is possible to construct a motion planning in the configuration space X so that the continuous path s(A, B) in X, which describes the movement of the system from the initial configuration A to the final configuration B, depends continuously on the pair of points (A, B) ?

Robotics & Topology

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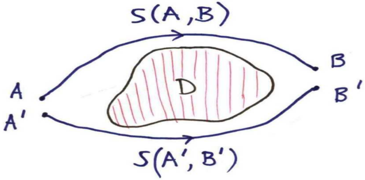
Topological interpretation

discontinuities, or instabilities, are dued to the topology of X.

Continuity of motion planning is an important natural requirement. Absence of continuity will result in the instability of behavior : there will exist arbitrarily close pairs (A, B) and (A', B') of initial-desired configurations such that the corresponding paths $s(A, B)$ and $s(A', B')$ are not close.



The first result

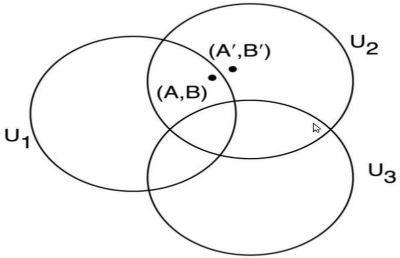
M. Farber (2003)

A continuous motion planning $s : X \times X \longrightarrow PX$ exists if and only if the configuration space X is contractible.

The premise

It is desirable to

minimise these discontinuities, to produce optimally stable MPAs. Discontinuity of the motion planner corresponding to a covering



The Founder (2003)

Michael Farber
University of Warwick
England

Definition

Given a path-connected topological space X, we define the topological complexity of the motion planning in X as the minimal number TC(X) = k, such that the Cartesian product $X \times X$ may be covered by k open and contracible subsets

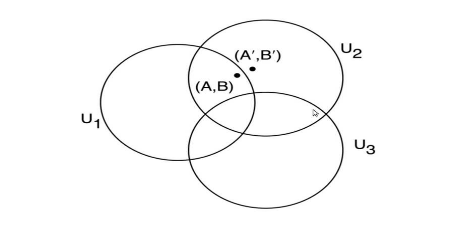
$$X \times X = U_1 \cup U_2 \cdots \cup U_k$$

That mean that for any $i = 1, 2, ..., k$ there exists a continuous motion planning $s_i : U_i \longrightarrow PX$. If no such k exists we will set $TC(X) = \infty$

Interpretation

Intuitively

the topological complexity TC(X) is the measure of discontinuity of any motion planner in X. Given an open cover and sections s_i as in the figure. Given a pair of initial-desired configurations (A, B) and suppose that (A, B) is close to the boundary of U_1 and to a pair $(A', B') \in U_2 - U_1$; then the output $s_1(A, B)$ compared with $s_2(A', B')$ may be completely different, since the sections $s_1|_{U_1 \cap U_2}$ and $s_2|_{U_1 \cap U_2}$ are in general distincts.



Equivalent version

M. Farber (2004)

Given a path-connected topological space X, we define the topological complexity of the motion planning in X as the minimal number TC(X) = k, such that the Cartesian product $X \times X$ may be covered by k disjoined nice ENR[a] subsets equipped with local continuos sections $s_i : F_i \longrightarrow PX$

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a. ENR : Euclidian Neighbourhood Retract

Homotopy Invariance

M. Farber(2003)

TC(X) depends only on the homotopy type of X.

Utility

This property of homotopy invariance often allows us to simplify the configuration space X without changing the topological complexity TC(X). Hence, we may predict the character of instabilities of the behavior of the robot knowing, for example, the cohomology algebra of its configuration space.

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Homotopy Invariance

Interpretation

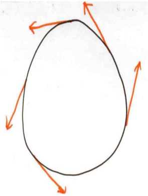
The instabilities in the robot motion planning algorithms depend on the homotopy properties of the robot's configuration space. In topology, if one regards the topological spaces as configuration spaces of mechanical systems, the homotopy invariant TC(X) is a new interesting tool which measures the "navigational complexity" of X.

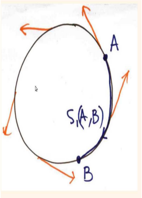
Inspiration

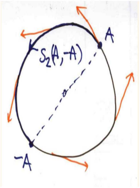
- The topological approach to the robot motion planning problem was initiated by M. Farber (2003 ; 2004).
- It was inspired by the earlier well-known work of Smale (1987) and Vassiliev (1988) on the theory of topological complexity of algorithms of solving polynomial equations.
- The approach of Farber (2003 ; 2004) was also based on the general theory of robot motion planning algorithms described in the book of J.-C. Latombe (1991).

Odd Spheres : $TC(\mathbb{S}^{2n}) = 2$, M. Farber (2003)

Spheres are not contractibles







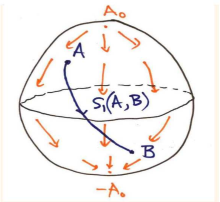
No vanishing vector Field v

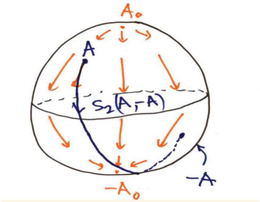
$F_1 = \{(A,B), B \neq -A\}$
s_1 = shortest path A to B

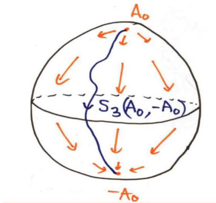
$F_2 = \{(A,A)\}$
s_2 = equator in direction v (A)

Even Spheres : $TC(\mathbb{S}^{2n+1}) = 3$, M. Farber (2003)

Vector field v with two zeroes $A_0, -A_0$







$F_1 = \{(A, B), B \neq -A\}$
s_1 = shortest path

$F_2 = \{(A, A), A \neq A_0, -A_0\}$
s_2 =positive equator

$F_3 = \{\pm(A_0, -A_0$
s_3 = any paths

Copie of n-spheres

M.Farber (2003)

Let $X = \mathbb{S}^m \times \cdots \times \mathbb{S}^m$ be a Cartesian product of n copies of the m-dimensional sphere $\mathbb{S}^m$. Then

$$
\begin{aligned}
TC(X) &= n+1 && \text{if m is odd} \\
&= 2n+1 && \text{if m is even.}
\end{aligned}
$$

Corollary : $TC(\mathbb{T}^n) = 2n + 1$

Open Problem (M. Grant)

$$TC(\text{Klein Bottle}) = 4 \text{ ou } 5?$$

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Flying Robot

Flying Robot

Flying robot in 3D

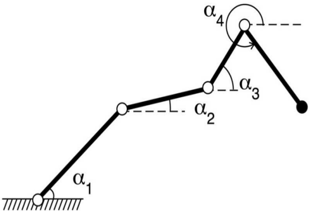
- The configuration space is $X = \underbrace{\mathrm{SO}(3)}_{\text{Orientation}} \times \underbrace{\mathbb{R}^3}_{\text{Position}}$.
- Since $X \simeq \mathrm{SO}(3)$ then $\mathrm{TC}(X) = \mathrm{TC}(\mathrm{SO}(3))$.
- We know form M. Farber, that for any topological group $\mathrm{TC}(X) = \mathrm{cat}(X)$
- It is known, that $\mathrm{cat}(\mathrm{SO}(3)) = 4$

navigation of a flying robot has a topological complexity equal to 4

Robot Arm

Robot arm

consisting of n bars $L_1, \ldots, L_n$, such that L_i and L_{i+1} are connected by flexible joins. In the planar case, a configuration of the arm is determined by n angles $\alpha_1, \ldots, \alpha_n$, where α_i is the angle between L_i and the x-axis



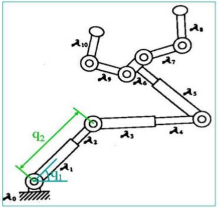
2D Robot Arm

In the planar case

the configuration space of the robot arm (when no obstacles are present) is the n-dimensional torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

Conclusion

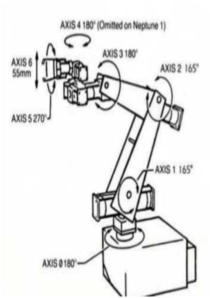
The topological complexity of the motion planning problem of a plane n-bar robot arm equals n+1



3D Robot Arm

Conclusion

The configuration space of a robot arm in the three-dimensional space $\mathbb{R}^3$ is the Cartesian product of n copies of the two-dimensional sphere $\mathbb{S}^2$.



The topological complexity of the 3D-motion planning problem of a spacial n-bar robot arm equals 2n + 1.

Other Complexities

Robot	Conf. Space
Mobile Robot translating in 2D	$\mathbb{R}^2$
Mobile Robot translating and rotating in 2D	$\mathbb{R}^2 \times \mathbb{S}^1 \simeq SE(2)$
Mobile Robot translating in 3D	$\mathbb{R}^3$
Spacecraft	$SO(3) \times \mathbb{R}^3$
n-joint revolute arm	T^n
2D mobile robot with n-joint arm	$SE(2) \simeq T^n$

That's all talks

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